Developing Intuition for Mathematical Reasoning

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Abstract

Logical intuition is fundamental to modern mathematics, enabling the perception of logical truth and resolution of mathematical problems. However, present-day AI systems lack intuitive mathematical reasoning, hampering performance. To imbue AI with mathematical intuition necessitates discussing the genesis of intuition and how it buttresses mathematical reasoning.

1 introduction

Envision becoming lost in a desert where every direction appears identical, forcing aimless wandering in search of an escape. Upon reaching a summit, a bird's-eye panorama reveals the terrain, facilitating navigation through this expanse. This is the power of intuition.

As defined by mathematicians, intuition encompasses the attitudes accrued through experience [11]. Cultivating intuition is regarded as integral to mathematical education; for instance, rather than explicitly directing students in procedural steps, instructors are advised to prompt independent thinking by inquiring "what should be done next?" [11].

Remark. *It should be noticed that this idea is close to Chain of Thought (CoT) [8], which can improve mathematical performance of Large Language Models (LLMs).*

In mathematical reasoning, intuition enables conjecturing and hypothesizing, guides proofs and calculations, and fuels inspiration and creativity. The interplay between intuitive senses and formal logic is integral to the practice of mathematics. Moreover, robust mathematical reasoning strategically employs both intuitive beliefs and logical arguments in tandem.

Mathematical reasoning poses profound challenges for AI systems, despite extensive research towards solving mathematical problems—with the International Mathematical Olympiad (IMO) representing the pinnacle of success. However, current AI performance remains unsatisfactory, constrained by near-infinite tactical possibilities. Consequently, imbuing logical intuition in AI systems is imperative.

In this essay, we elucidate how intuition can develop in AI systems and how it buttresses mathematical reasoning.

2 Develop Intuition

> Young man, in mathematics you don't understand things. You just get used to them.
>
> *John von Neumann*

As von Neumann [10] articulated, comprehending mathematics does not stem from eureka moments, but rather the accretion of expansive experience - advancing through persistence and habituation. Mathematical intuition thus germinates by cultivating an innate, nearly subconscious sense for the subject that stems directly from this process.

To engender intuition within AI systems, we ought to explore the genesis of human intuition and endeavor to simulate analogous mechanisms computationally. Herein, we introduce two integral components undergirding logical intuition: **analogy** and **number sense**.

2.1 Develop an Analogical Engine

Analogies emerge amid commonalities between cases [17]. As an intuitional capacity, analogical reasoning fulfills a pivotal role within mathematical cognition by facilitating the discernment and transference of applicable knowledge from resolved precedents. This scaffolds solving nascent problems by subsidizing recognition of salient features and tactical associations. Constructing an analogical engine could thus enable AI math solvers to reason more intuitively and adaptively by capitalizing on accrued experience.

Analogical reasoning ability has undergone scrutiny in another domain of cognitive science. Raven's Progressive Matrices (RPM) stands out as a well-established assessment tool for abstract reasoning and fluid intelligence. The test presents a configuration of visual geometric figures, with one element missing. Test-takers are tasked with intuitively discerning the underlying pattern and selecting the appropriate analogical figural completion that aligns with the pattern.

The tasks involving abstract visual pattern completion demonstrate a structural similarity to the recognition of abstract relational patterns found in mathematical theorems and proofs. Notably, studies indicate a close relationship between Raven's matrices and mathematical problem-solving [7].

In the investigation of analogy-making ability, Zhang et al. [13] constructed the RAVEN dataset using an Attributed Stochastic Image Grammar (A-SIG), as illustrated in Fig. 1.

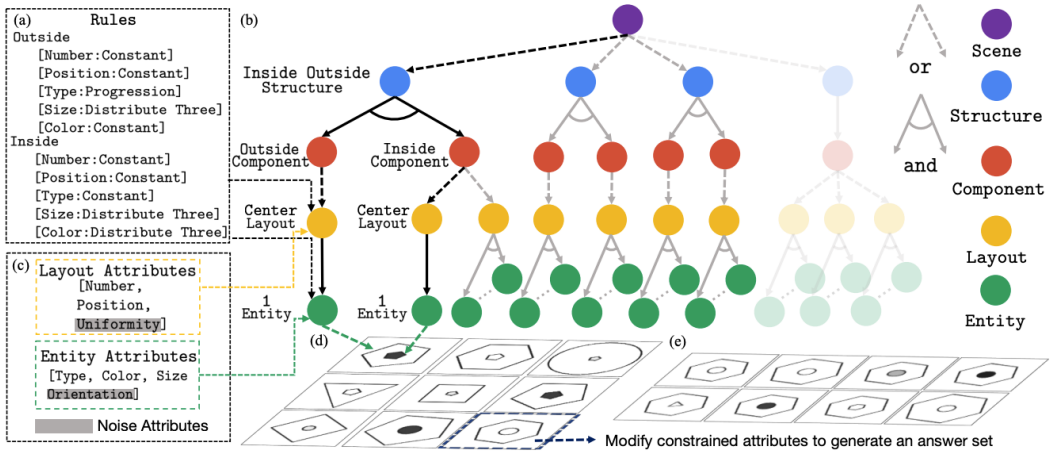


Figure 1: RAVEN creation process. Image credit: Zhang et al. [13].

Efforts have been devoted to discerning the structured features inherent in Raven's Matrices. [13] introduced a tree-structured residual module termed Dynamic Residual Tree (DRT) to capture these structured features. Furthermore, [14] proposed a neuro-symbolic Probabilistic Abduction and Execution (PrAE) learning approach, effectively disentangling the processes of perception and reasoning. Additionally, in their work, [15] employed a similar structure, explicitly modeling algebraic calculations to facilitate systematic generalization.

While methods for solving Raven's Progressive Matrices (RPM) have demonstrated commendable performance, their generalization ability remains constrained. Uncovering analogies between a novel mathematical problem and pre-existing solutions necessitates the development of a representational scheme capable of encoding proofs and traces of problem-solving as structured knowledge. Additionally, an efficient search algorithm is essential for identifying the most pertinent analogies.

2.2 Develop Number Sense

The development of number sense has been deemed crucial in early education. Number sense cultivates a profound understanding of the "essence" of numbers, quantitative relationships, and numerical transformations. A robust number sense facilitates seamless mental manipulation of

numbers, enabling heuristic reasoning in dealing with quantities, such as employing estimation strategies. For instance, it heavily relies on the ability to estimate in order to navigate inequalities.

Moreover, pattern recognition, stemming from a well-honed number sense, enables the discernment of regularities in calculations, algebraic expansions, series relationships, and more. A robust intuitive "number sense" lays a vital foundation for mathematical reasoning. The close relationship between an intuitive sense of number and formal mathematical ability is asserted in the literature [3]. By incorporating capabilities akin to number sense, potentially inspired by cognitive models in child development, AI systems can significantly enhance their intuitive reasoning prowess in mathematical tasks.

In the assessment of the number sense of AI systems, Zhang et al. [16] employed an And-Or Graph (AOG) to construct the Machine Number Sense (MNS) dataset. Utilizing visual arithmetic problems as opposed to language-based ones, they evaluated the performance of state-of-the-art (SOTA) neural networks and search-based models.

3 Guide AI math solver with intuition

This section explores the role of mathematical intuition, focusing on analogical reasoning and number sense, in enhancing AI math solvers. Two primary methods, namely the language model prover and the search-based prover, will be employed as illustrative examples.

3.1 Language Model Prover

The advent of transformer-based language models has led to the proposal of various methods for solving mathematical problems, such as GPT-f [6], NaturalProver [9], and others. However, theorem proofs necessitate premises that must be retrieved from a knowledge base. Yang et al. [12] introduce the Retrieval-Augmented Prover (ReProver). Their retriever is built upon Dense Passage Retrieval (DPR) and generates tactics conditioned on the retrieved proofs.

The retrieval process shares similarities with analogical reasoning. Analogical matching plays a pivotal role in surfacing the most structurally relevant proofs or theorems from the retrieval set, enabling their application to the context of a new problem. The identification of analogies proves valuable in pruning fruitless proof paths by recognizing dead-end patterns based on prior experience. Additionally, number sense facilitates intuitive manipulation of numbers and quantities when mapping proofs between analogical problems.

3.2 Search-based Prover

The process of theorem proving can be conceptualized as a tree-generating procedure, where different tactics lead to distinct branches. Techniques such as alpha-beta pruning [4] and Monte Carlo Tree Search (MCTS) [1] are employed to prune the search space. However, these methods grapple with the combinatorial explosion of tactics, stemming from the infinite possibilities of moves.

An impressive model inspired by MCTS, proposed by Lample et al. [5], demonstrates remarkable performance by successfully solving 10 IMO problems. This model incorporates both a policy model and a critic model to guide the search process. Despite its notable achievements, proving entire datasets remains challenging, and issues such as data insufficiency persist [5].

Mathematical intuition plays a pivotal role in guiding the search process through various avenues. Number sense facilitates the rapid estimation of the "value" of subsequent expansion steps, thereby enhancing the selection policy. Analogical evaluation of branch options can fortify the learned evaluation model, identifying the most promising paths. Furthermore, analogical hints derived from memory contribute to directing tree growth by recognizing unexplored yet recursively related branches.

Incorporating the pattern recognition capabilities from both analogical and number sense subsystems would enable more informed expansion policies, underpinned by intangible experiential knowledge, when navigating the intricate proof search spaces. Such integration has the potential to significantly enhance proof-finding efficiency. The key lies in seamlessly blending and unifying the more logic-based search mechanisms with the more loosely intuition-directed processes to achieve optimal

results. In this manner, the collective prowess of AI in mathematical problem-solving becomes greater than the sum of its individual components.

4 Conclusion

In summary, we have explored the development and role of intuition in mathematical reasoning. While methods have been proposed to augment human mathematicians with AI "intuition" [2], the realization of a fully autonomous AI mathematician remains a distant prospect. In addition to intuition, the incorporation of logic-based mathematical knowledge into AI is essential, encompassing elements such as axioms, definitions, theorems, lemmas, and propositions coded within a symbolic knowledge base.

Furthermore, the integration of libraries containing diverse mathematical problems is crucial for training reinforcement learning and neural network-based approaches. Exposure to a variety of problems can guide the gradual development of intuitive skills. It is our hope that this essay serves as inspiration for further research on this intriguing topic.

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