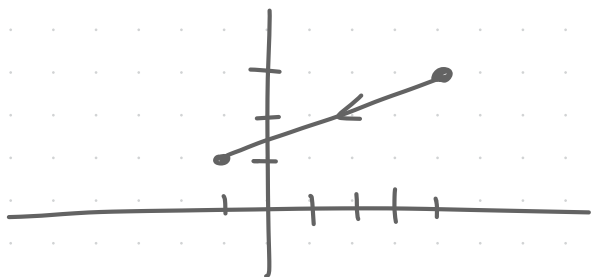


Parametrizing a line segment

ex: from $(4, 3)$ to $(-1, 1)$



$$\begin{cases} x(t) = ? \\ y(t) = ? \end{cases} \\ 0 \leq t \leq 1$$

Want: $x(0) = 4$
 $y(0) = 3$

$$\begin{aligned} x(1) &= -1 \\ y(1) &= 1 \end{aligned}$$

$$\begin{cases} x(t) = 4 - 5t \\ y(t) = 3 - 2t \end{cases}$$

$$0 \leq t \leq 1$$

← goes left by 5

← goes down 2

$\xrightarrow{F}$
|=====|

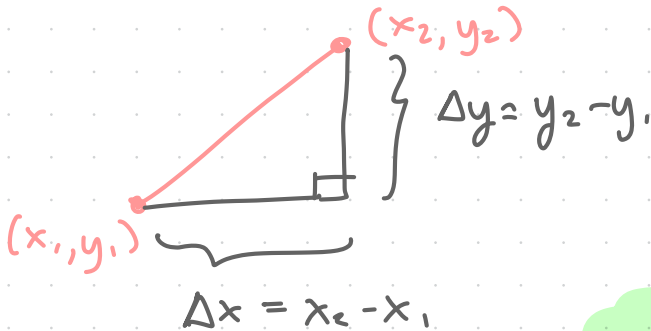
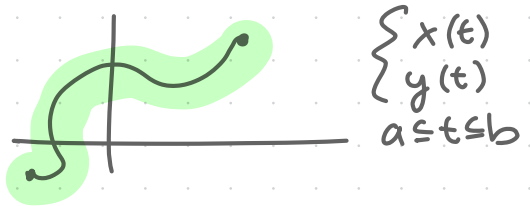


$$\text{work: } \int_0^{0.1} F dx$$

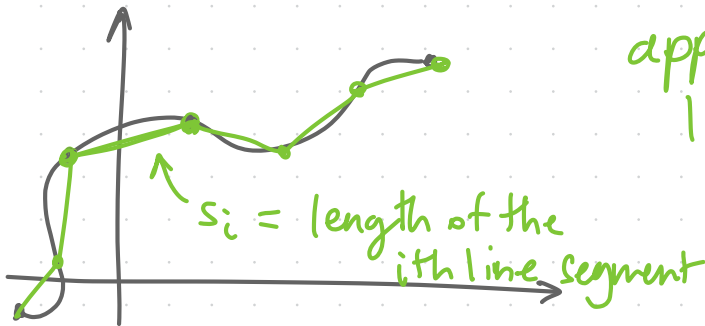
$$F(x) = kx$$

6.4 - Arc Length

Recall the distance formula:

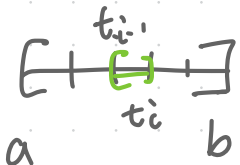


$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

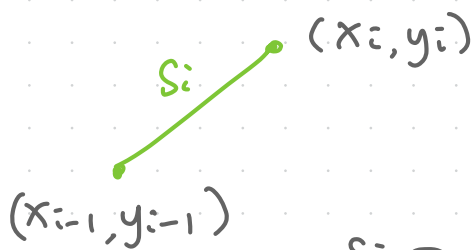


approx. with
line segments

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \quad a \leq t \leq b$$



$$\text{length} \approx \sum s_i$$



$$S_i = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

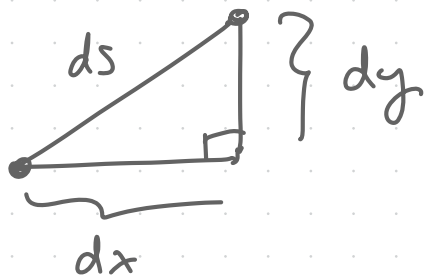
$$= \sqrt{(f(\underline{t_i}) - f(\underline{t_{i-1}}))^2 + (g(\underline{t_i}) - g(\underline{t_{i-1}}))^2}$$

How to smooth this out?

take a limit as $\Delta t \rightarrow 0$.

Δt changes into dt

Consider an infinitesimal line segment of length ds :



$$\text{length} = ds$$

$$= \sqrt{(dx)^2 + (dy)^2}$$

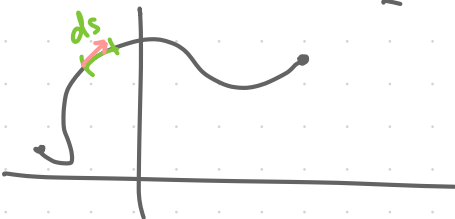
$$x = f(t) \quad \Rightarrow \quad dx = f'(t) dt$$

$$y = g(t) \quad \Rightarrow \quad dy = g'(t) dt$$

$$\begin{aligned} \text{So } ds &= \sqrt{(dx)^2 + (dy)^2} \\ &= \sqrt{(f'(t) dt)^2 + (g'(t) dt)^2} \\ &= \sqrt{f'(t)^2 dt^2 + g'(t)^2 dt^2} \\ &= \sqrt{f'(t)^2 + g'(t)^2} \underbrace{\sqrt{dt^2}}_{dt} \\ &= \underbrace{\sqrt{f'(t)^2 + g'(t)^2}}_{dt} dt \end{aligned}$$

$$\text{total length} = \int_a^b ds$$

$$= \int_a^b \sqrt{\underline{f'(t)^2} + \underline{g'(t)^2}} dt$$



Arc Length Formula: (for parametric curves)

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Important: the curve must be traversed exactly once to give the correct answer

length not distance travelled

example:
$$\begin{cases} x = t^2 \\ y = t^3 \end{cases}$$

$$0 \leq t \leq 2$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= \sqrt{(2t)^2 + (3t^2)^2} dt$$

$$= \sqrt{4t^2 + 9t^4} dt$$

$$= \sqrt{t^2 \cdot (4 + 9t^2)} dt$$

$$= \sqrt{t^2} \cdot \sqrt{4 + 9t^2} dt$$

$$= |t| \sqrt{4 + 9t^2} dt = t \sqrt{4 + 9t^2} dt$$

$$L = \int_0^2 ds$$

$$= \int_0^2 t \sqrt{4+9t^2} dt$$

$$= \int_4^{40} \sqrt{u} \cdot \frac{1}{18} du$$

$$= \frac{1}{27} (40^{3/2} - 4^{3/2})$$

$$\begin{aligned} u &= 4 + 9t^2 \\ du &= 18t dt \end{aligned}$$

$$t=0 \Rightarrow u=4$$

$$t=2 \Rightarrow u=40$$

$$\frac{1}{18} \cdot \frac{2}{3} = \frac{1}{27}$$

usual

3 variations:

u-sub

trig sub

algebra & get a perfect square
inside the square root

ex: $\begin{cases} x = e^t - t \\ y = 4e^{t/2} \end{cases}$
 $-8 \leq t \leq 0$

$$\frac{d}{dt} (e^{\frac{1}{2}t}) = \frac{1}{2} e^{\frac{1}{2}t}$$

$$\begin{aligned} \underline{ds} &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \underline{dt} \\ &= \sqrt{(e^t - 1)^2 + (2e^{\frac{1}{2}t})^2} dt \end{aligned}$$

$$= \sqrt{(e^t - 1)^2 + 4 \cdot e^t} dt$$

$$= \sqrt{(e^{2t} - 2e^t + 1) + 4e^t} dt$$

$$= \sqrt{e^{2t} + 2e^t + 1} dt$$

$$(e^t - 1)^2 = e^{2t} - 2e^t + 1$$

↑ compare to

$$(e^t + 1)^2 = e^{2t} + 2e^t + 1$$

$$= \sqrt{(e^t + 1)^2} dt$$

$$= e^t + 1 dt$$

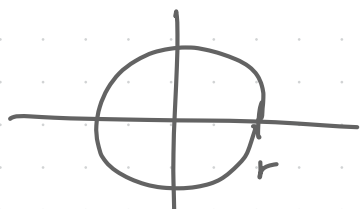
$$L = \int_{-8}^0 e^t + 1 dt = (e^t + t) \Big|_{-8}^0$$

$$= (e^0 + 0) - (e^{-8} - 8)$$

$$= 1 - e^{-8} + 8$$

$$= 9 - e^{-8}$$

ex: Show the circumference of a circle = $2\pi r$



parametrize:

$$\begin{cases} x = r \cos t \\ y = r \sin t \end{cases}$$

$$0 \leq t \leq 2\pi$$

↑
radius r

↑
goes around
once

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= \sqrt{(-r \sin t)^2 + (r \cos t)^2} dt$$

$$= \sqrt{r^2 \sin^2 t + r^2 \cos^2 t} dt$$

$$= \sqrt{r^2 \underbrace{(\sin^2 t + \cos^2 t)}_1} dt$$

$$= r dt$$

$$\begin{aligned}
 L &= \int_0^{2\pi} ds = \int_0^{2\pi} r \, \underline{dt} \\
 &= (r t) \Big|_0^{2\pi} \\
 &= 2\pi r \quad \checkmark
 \end{aligned}$$

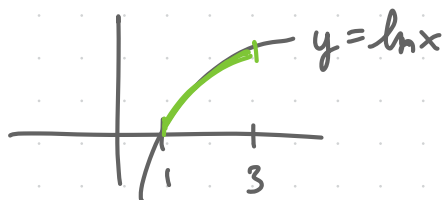
Computing the length of a curve given as a function of x or function of y . (not a param. curve)

ex: $y = \ln x$ where $1 \leq x \leq 3$

$$\begin{aligned}
 \text{parametrize: } &\begin{cases} x = t \\ y = \ln t \end{cases} \\
 &1 \leq t \leq 3
 \end{aligned}$$

$$\begin{aligned}
 ds &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
 &= \sqrt{1^2 + \left(\frac{1}{t}\right)^2} dt \\
 &= \sqrt{1 + \frac{1}{t^2}} dt
 \end{aligned}$$

$$L = \int_1^3 \sqrt{1 + \frac{1}{t^2}} dt \quad \left. \vphantom{\int_1^3} \right\} \text{hard integral, so just set up}$$



In general, for a curve given as $y = f(x)$ for $a \leq x \leq b$, can always parametrize as

$$\left. \begin{array}{l} x = t \\ y = f(t) \\ a \leq t \leq b \end{array} \right\}$$

$$ds = \sqrt{1 + f'(t)^2} dt$$

$$= \sqrt{1 + f'(x)^2} dx \quad \text{or} \quad \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

← Arc length formula

If given a curve as $x = g(y)$ for $a \leq y \leq b$,
can always parametrize as:

$$\begin{cases} x = g(t) \\ y = t \end{cases}$$

$$a \leq t \leq b$$

$$\Rightarrow ds = \sqrt{g'(t)^2 + 1} dt$$

$$= \sqrt{g'(y)^2 + 1} dy$$

$$= \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} dy$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} dy$$

ex: Find the length of

$$y = \ln(\sec x)$$

$0 \leq x \leq \pi/3$

$$y = \ln(\sec x)$$

$$\frac{dy}{dx} = \frac{1}{\sec x} \cdot \sec x \tan x$$

function of x ,
not a param.
Curve



use

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$\text{Use: } ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= \sqrt{1 + \left(\frac{1}{\cancel{\sec x}} \cdot \cancel{\sec x} \tan x\right)^2} dx$$

$$= \sqrt{1 + \tan^2 x} dx$$

$$\frac{\sin^2 x + \cos^2 x = 1}{\cos^2 x} \quad \frac{1}{\cos^2 x}$$

$$\tan^2 x + 1 = \sec^2 x$$

$$= \sqrt{\sec^2 x} dx$$

$$= |\sec x| dx$$

$$= \sec x dx$$

$$\left(\cos x \geq 0 \text{ on } [0, \pi/3] \right)$$

so $\sec x \geq 0$

$$L = \int_0^{\pi/3} ds = \int_0^{\pi/3} \sec x \, dx$$

$$= \left(\ln |\sec x + \tan x| \right) \Big|_0^{\pi/3}$$

$$= \ln \left| \sec\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{3}\right) \right| - \ln |\sec(0) + \tan(0)|$$

$$= \ln |2 + \sqrt{3}| - \ln \left| \frac{1}{\cos(0)} + 0 \right|$$

$$= \ln |2 + \sqrt{3}| - \cancel{\ln(1)}^{\circ}$$

$$= \ln(2 + \sqrt{3})$$

