HOMOLOGY AND SOME APPLICATIONS

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3-Manifolds Seminar

Departamento de Matemática - ICEx - UFMG

14/12/2020

Contents

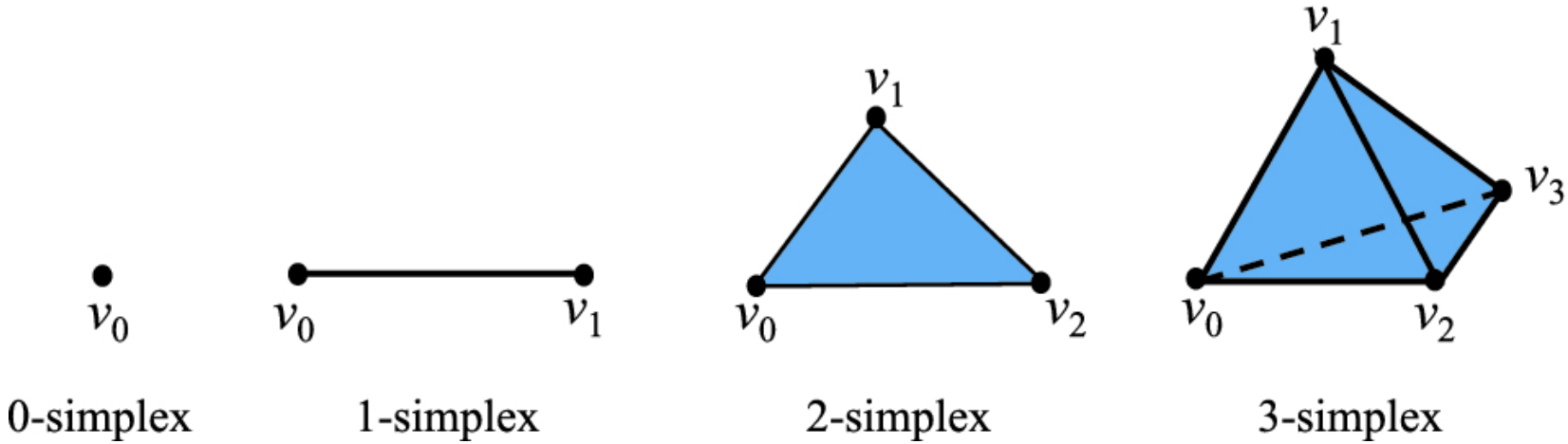
Homology

The Boundary

Non orientable surfaces in 3-Manifolds

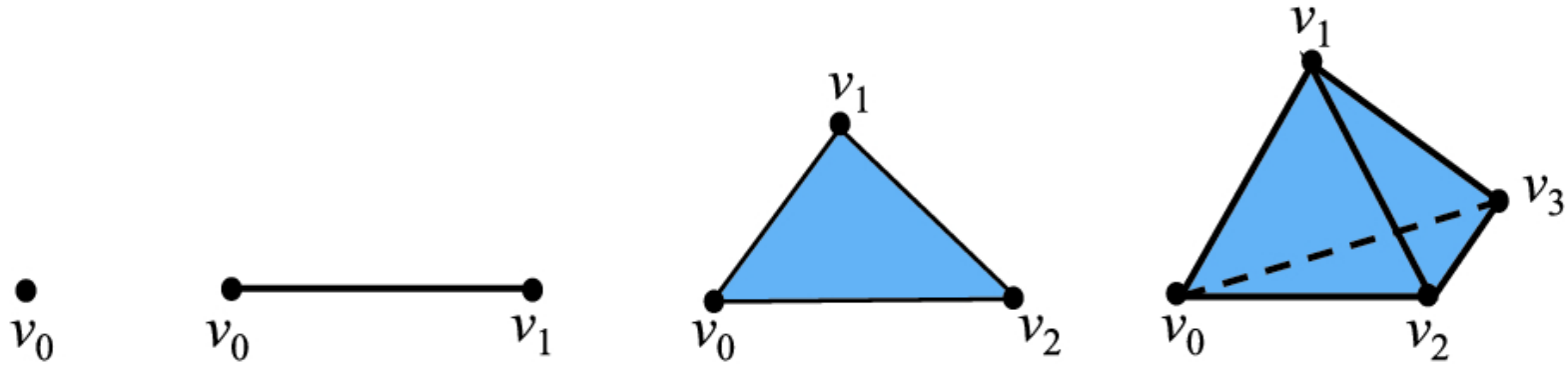
Simplicial Homology I

- k-simplex Δ^k:



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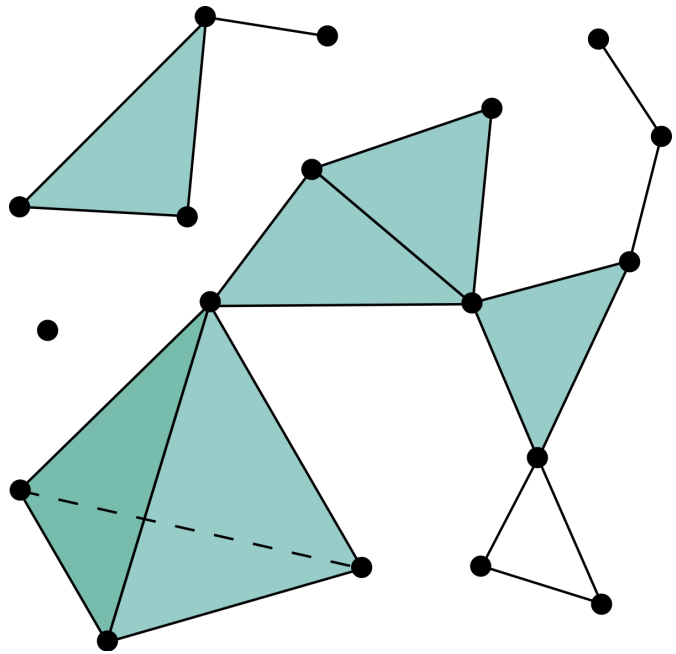
- A k-chain is a finite formal sum

$$\sum_i \lambda_i \Delta_i^k,$$

where $\lambda_i \in \mathbb{Z}$ and Δ_i^k a k-simplex.

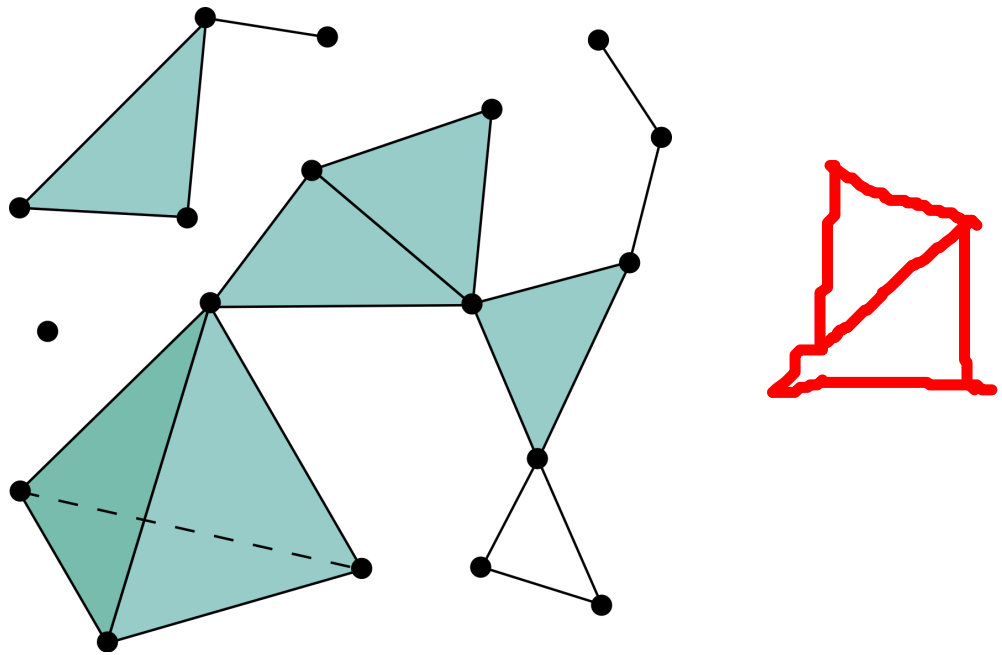
Simplicial Homology II

- Simplicial complex:
  1. Finite set $\mathcal{K}$ of simplices such that every face of a simplex in $\mathcal{K}$ is also in $\mathcal{K}$.
  2. Every non-empty intersection of two simplices is a common face.



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- Geometric realization: Homeomorphism with a topological space.
- This is called triangulation.
- Smooth manifolds are always triangularizable.

Simplicial Homology III

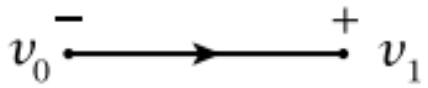
- $C_k(X, R)$ denotes the set of all k-chains.
- $\partial_k : C_k(X, R) \longrightarrow C_{k-1}(X, R)$ is the boundary operator defined by

$$\partial_k(\alpha) = \sum_i (-1)^i [v_0, ..., \hat{v}_i, ..., v_n].$$

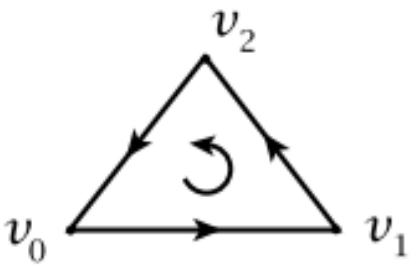
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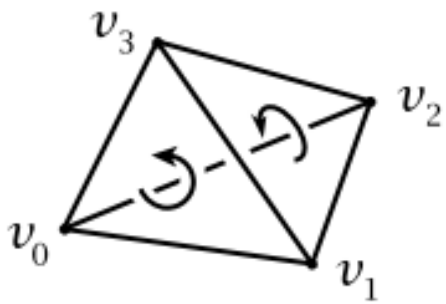
$$\partial_k(\alpha) = \sum_i (-1)^i [v_0, ..., \hat{v}_i, ..., v_n].$$



$$\partial[v_0, v_1] = [v_1] - [v_0]$$



$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$



$$\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$$

- $\partial_k \circ \partial_{k+1} = 0$.
- The elements of $\operatorname{Ker} \partial_k$ are called cycles.
- The elements of $\operatorname{Im} \partial_{k+1}$ are called boundaries.
- The k-th simplicial homology group is

$$H_k(X, \mathbb{Z}_2) = \operatorname{Ker} \partial_k / \operatorname{Im} \partial_{k+1},$$

Singular Homology I

- Let X be a topological space and R a ring.
- A singular k-simplex is a continuous map $\alpha\colon \Delta^k \longrightarrow X$.
- A k-chain is a finite formal sum

$$\sum_i \lambda_i \alpha_i,$$

where $\lambda_i \in R$ and α_i a singular k-simplex.

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- The k-th homology group is

$$H_k(X, R) = \operatorname{Ker} \partial_k / \operatorname{Im} \partial_{k+1},$$

- The dual spaces $C_k(X, R)^*$ and the dual operator ∂_k^* are considered to define the k-th cohomology group

$$H^k(X, R).$$

Properties

- Let X be path-connected, then $H_0(X, R) = R$.

Properties

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- There is a canonical homomorphism

$$\pi_1(X) \longrightarrow H_1(X, \mathbb{Z})$$

  which maps a class $[\lambda]$ to the cycle λ.
- Note that $\partial\lambda = 0$.
- $H_1(X, \mathbb{Z})$ is the abelianization of $\pi_1(M)$.
- $H^1(X, \mathbb{Z}) = \mathrm{Hom}(H_1(X, \mathbb{Z})) = \mathrm{Hom}(\pi_1(X), \mathbb{Z})$.

Continuation

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- $H_k(M, \mathbb{Z})$ is finitely generated (Existence of triangulation).
- $H_k(M, \mathbb{Z}) \simeq \mathbb{Z}^{b_k} \oplus T_k$ (Theorem of finitely generated abelian groups).
- $H^k(M, \mathbb{Z}) \simeq \mathbb{Z}^{b_k} \oplus T_{k-1}$, T_i is the torsion.

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- $H^k(M, \mathbb{Z}) \simeq \mathbb{Z}^{b_k} \oplus T_{k-1}$, T_i is the torsion.
- If $\partial M = \emptyset$, there is the Poincaré duality:

$$H^k(M, R) \simeq H_{n-k}(M, R).$$

Application

Proposition

Let M be a closed orientable 3-manifold. The homology groups $H_*(M, \mathbb{Z})$ are determined by $\pi_1(M)$.

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Proof.

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- $H_0(M, \mathbb{Z}) = \mathbb{Z}$ (null torsion).
- $H_1(M) = F_1 \oplus T_1$

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- $H_0(M, \mathbb{Z}) = \mathbb{Z}$ (null torsion).
- $H_1(M) = F_1 \oplus T_1$
- $H^1(M) = F_1 \oplus T_0$, then $H^1(M)$ is determined.
- $H_2(M, \mathbb{Z}) = H^1(M)$, by Poincaré duality.
- $H_3(M, \mathbb{Z}) = H^0(M, \mathbb{Z}) = \mathbb{Z}$.

□

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Euler Characteristic

Let X be a finite CW-complex. The Euler characteristic is

$$\chi(X) = \sum_{i=0}^{n} (-1)^i c_i,$$

where c_i is the number of i-cells in X. It holds that

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Proposition

Let M be a closed odd dimensional manifold. Then $\chi(M) = 0$.

Oriented case.

- $H^k(M, \mathbb{Z}) = H_{n-k}(M, \mathbb{Z})$, since M is closed and oriented.
- In particular, $b_i = b_{n-i}$, by decomposition in cyclic groups.
- The sum vanishes.

Euler Characteristic of the boundary

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Let M be a compact odd dimensional manifold. Then $\chi(\partial M) = 2\chi(M)$.

- M orientable and $\partial M = \emptyset$: The previous result and $\chi(\partial M) = 0$.

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- M non-orientable and $\partial M = \emptyset$: The orientable double cover N has odd dimension and

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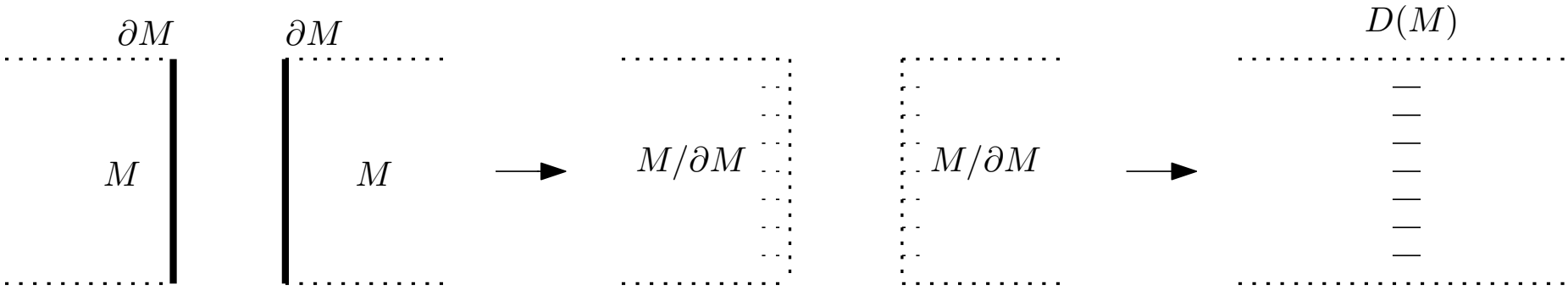
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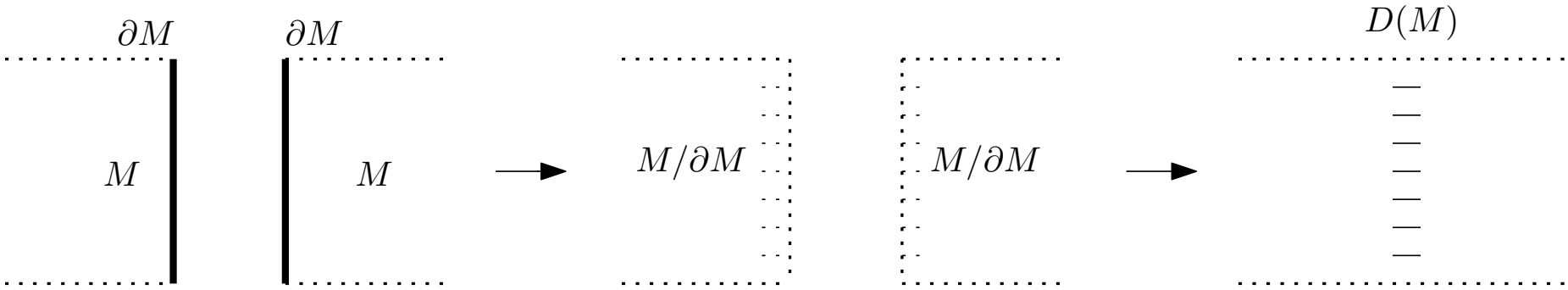
$$\chi(N) = d\chi(M).$$

- $\partial M \neq \emptyset$: Construct a double copy $D(M)$ of M identifying the boundary:

Continuation



Continuation



- $\dim(D(M)) = \dim(M)$
- $\partial(D(M)) = \emptyset$.
- $0 = \chi(D(M)) = 2\chi(M) - \chi(\partial M)$.

□

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Orientability

- A connected compact manifold M is orientable iff

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- The fundamental class $[M]$ is a choice of a generator of $H^0(M, \mathbb{Z})$.

Orientability

- A connected compact manifold M is orientable iff

$$H^0(M, \mathbb{Z}) \simeq \mathbb{Z}.$$

- The fundamental class $[M]$ is a choice of a generator of $H^0(M, \mathbb{Z})$.
- It induces a local orientation (local and consistent choice of generators).
- R-orientation: generator of $H^0(M, \mathbb{Z}) \simeq R$.
- Every manifold is $\mathbb{Z}_2$-orientable.

Continuation

Theorem

Let M be a closed connected manifold. Then:

1. *If M is R-orientable, $H_n(M, R) = R$.*

2. $H_i(M, R) = 0$ *for* $i > n$.

Normal Bundle

Definition

Let $N \subset M$ be a submanifold. The normal space at $p \in N$ is the quotient

$$\nu_p N = T_p M / T_p N.$$

The normal bundle is

$$\nu N = \bigcup_{p \in N} \nu_p N.$$

It has a structure of vector bundle.

Tubular Neighborhood

Definition

Let $N \subset M$ a submanifold, $m = dim(M)$, and $n = dim(N)$. A tubular neighborhood of N in M is an embedding of the normal bundle νN of N into M, where the image of the zero section is equal to N.

Continuation

- If N is a closed smooth submanifold, there exists a tubular neighborhood.

- A fibre bundle $\pi : E \longrightarrow B$ is a line bundle if $\pi^{-1}(p) \simeq \mathbb{R}$ for all $p \in B$.

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- A fibre bundle $\pi : E \longrightarrow B$ is a line bundle if $\pi^{-1}(p) \simeq \mathbb{R}$ for all $p \in B$.

Proposition

A connected compact manifold M has a unique orientable line bundle E which is unique up to isomorphism.

If M is orientable, $E = M \times \mathbb{R}$. Otherwise, $M \tilde{\times} \mathbb{R}$ denotes the line bundle E.

Surfaces

- S k-dimensional orientable and closed defines a class in $H^0(M, \mathbb{Z}) = H_k(M, \mathbb{Z})$.

- S non-orientable defines a class in $H_k(M, \mathbb{Z}_2)$.

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- S non-orientable defines a class in $H_k(M, \mathbb{Z}_2)$.

Let M be a connected orientable 3-manifold and $S \subset M$ a closed connected surface

- Any tubular neighborhood of S (orientable) is diffeomorphic to $E = S \times \mathbb{R}$.

- Similarly for a non-orientable S and $E = S\tilde{\times}\mathbb{R}$.

Intersection

- Let M be a compact oriented n-manifold with (p. empty) boundary.
- S, S' oriented closed k, $n-k$-submanifolds, respectively.
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- Orientability of $T_xS \oplus T_xS'$ and T_xM. If agree or not, assign ± 1.
- $S \cdot S'$ denotes the sum of signs.

Theorem

Let $S.S'$ be transverse with classes $[S] \in H_k(M)$ and $[S'] \in H^{n-k}(M)$. Then

$$\omega([S],[S']) = S \cdot S'.$$

Non-orientable case

- The ring considered is $\mathbb{Z}_2$.
- Similar results hold.
- We may consider classes in $H_k(M, \mathbb{Z}_2)$.
- Signs taken mod 2.

Non-orientable surfaces in 3-manifolds

Proposition

Let M be a compact oriented manifold with (p. empty) boundary. Every non-orientable surface determines a non-trivial class $[S] \in H^1(M, \mathbb{Z}_2)$. The manifold M can not contain more than $dim\, H^1(M, \mathbb{Z}_2)$ disjoint non-orientable surfaces.

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Proof.

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Proof.

- Since S is closed, there exists a tubular neighborhood $E = S\tilde{\times}\mathbb{R}$.
- There is a lifting $\alpha \in E$ of a closed simple loop in S (Lifting property for fibre bundles).
- α intersects S at one point.

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- α intersects S at one point.
- $\alpha \cdot S = \omega(\alpha, S) = 1$.

□

Continuation

- $[S]$ is a non trivial class in $H^1(M, \mathbb{Z}_2)$.
- Each non-orientable surface S_i has a transverse simple closed loop α_i.
- The elements $[S_1], ..., [S_k]$ are linearly independent in $H^1(M, \mathbb{Z}_2)$.

□

Continuation

Corollary

A simply-connected compact 3-manifold with (p. empty) boundary does not contain any closed non-orientable surface.

Proof.

- $\pi_1(M)$ is trivial.
- By the universal coefficient theorem,

$$H^1(X, G) = \mathrm{Hom}(H_1(X), G),$$

where G is abelian.

Continuation

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Proof.

- $\pi_1(M)$ is trivial.
- By the universal coefficient theorem,

$$H^1(X, G) = \mathrm{Hom}(H_1(X), G),$$

  where G is abelian.
- If X is path-connected,

$$H^1(X, G) = \mathrm{Hom}(\pi_1(X), G).$$

- $\mathrm{Hom}(\pi_1(X), \mathbb{Z}_2) \simeq 0$.
- The result follows from the previous proposition.

□

OBRIGADO!