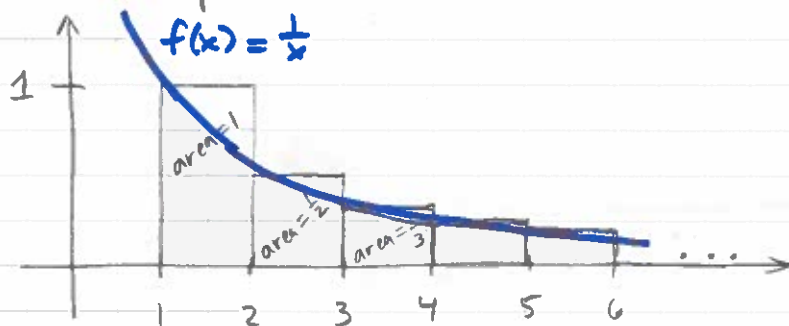


### 11.3 - The Integral Test

Two motivating examples, then we'll state the test.

Example 1:  $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

We'll represent this series with rectangles:



sum of areas of rectangles =  $\sum_{n=1}^{\infty} \frac{1}{n}$

Add the graph of  $f(x) = \frac{1}{x}$  to this picture.

Recall that  $\int_1^{\infty} \frac{1}{x} dx$  diverges (to infinity)

Also, see from the picture that

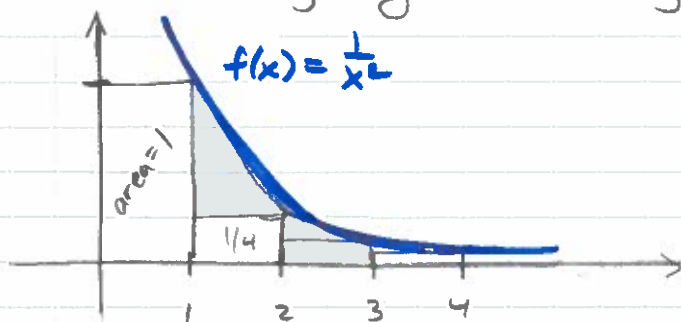
$$\left( \text{area under } f \text{ on } [1, \infty) \right) < \left( \text{sum of area of rectangles} \right)$$

$$\int_1^{\infty} \frac{1}{x} dx < \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

Example 2:  $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots$

We'll also represent this series with rectangles, but slightly differently:



sum of areas of rect's  $= \sum_{n=1}^{\infty} \frac{1}{n^2}$

(Add the graph of  $f(x) = \frac{1}{x^2}$  to this picture.)

Recall that  $\int_1^{\infty} \frac{1}{x^2} dx$  converges and its value is 1.

From the picture,

$$\int_1^{\infty} \frac{1}{x^2} dx > \text{sum of area of the rectangles (starting from 2nd rect.)}$$

$$1 > \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$$

$$\underline{+ 1}$$

$$\underline{+ 1}$$

$$2 > 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Claim: If  $\{S_n\}$  is the sequence of partial sums of  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ , then  $\{S_n\}$  converges

(Means the series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges)

$$S_n = 1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{n^2}$$

Notice:

- Each  $S_n$  is positive since  $S_n$  is a sum of positive terms  $\Rightarrow \{S_n\}$  is bounded below by 0.
- Since we only add positive terms,  $\{S_n\}$  is increasing:  
 $S_1 < S_2 < S_3 < S_4 < \dots$

- As we saw geometrically,

$$S_n < \lim_{n \rightarrow \infty} S_n = \sum_{n=1}^{\infty} \frac{1}{n^2} < 2$$

$\Rightarrow \{S_n\}$  is bounded above by 2.

- the sequence  $\{S_n\}$  converges by the Monotone Convergence Theorem.

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges.}$$

or recap after activity

### The Integral Test

Given a series  $\sum_{n=k}^{\infty} a_n$ , if  $f$  is a function such that  $f(n) = a_n$  for each  $n \geq k$  and

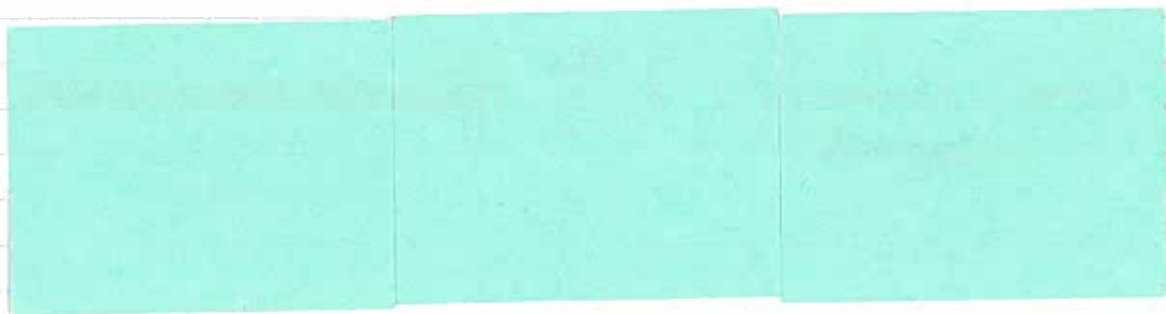
- $f$  is continuous
  - $f$  is positive
  - $f$  is decreasing
- } on  $[k, \infty)$

then  $\sum_{n=k}^{\infty} a_n$  converges if and only if  $\int_k^{\infty} f(x) dx$  converges.

This is saying 2 things:

① If  $\int_k^{\infty} f(x) dx$  converges, then  $\sum_{n=k}^{\infty} a_n$  converges

② If " " diverges, " " diverges.



example:  $\sum_{n=1}^{\infty} n e^{-n^2}$  converges or diverges?

Use the Integral Test. Let  $f(x) = x e^{-x^2}$

Check the 3 conditions:

- $f$  is continuous because it's a product of continuous functions ✓

- $f$  is positive because  $e^{-x^2} > 0$  for all  $x$  and  $x > 0$  for  $x$  in  $[1, \infty)$  ✓

- $f$  decreasing on  $[1, \infty)$ ?

$$f'(x) = 1 \cdot e^{-x^2} + x(-2x e^{-x^2}) \\ = e^{-x^2}(1 - 2x^2)$$

$$f'(x) = 0 \quad \text{if} \quad 1 - 2x^2 = 0 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$$



| $x$ | sign of $f'$ |
|-----|--------------|
| -10 | -            |
| 0   | +            |
| 10  | -            |

$f$  is decreasing on  $[\frac{1}{\sqrt{2}}, \infty)$

Since  $\frac{1}{\sqrt{2}} < 1$ , means  $f$  is decreasing on  $[1, \infty)$  ✓

The 3 conditions are satisfied so we can apply the Integral Test.

$$\begin{aligned}\int_1^{\infty} x e^{-x^2} dx &= \lim_{t \rightarrow \infty} \int_1^t x e^{-x^2} dx & u &= x^2 \\ & & du &= 2x dx \\ &= \lim_{t \rightarrow \infty} \int_1^{t^2} e^{-u} \cdot \frac{1}{2} du & x=1 &\Rightarrow u=1 \\ & & x=t &\Rightarrow u=t^2 \\ &= \frac{1}{2} \lim_{t \rightarrow \infty} (-e^{-u}) \Big|_1^{t^2} \\ &= \frac{1}{2} \lim_{t \rightarrow \infty} (-e^{-t^2} - (-e^{-1})) \\ &= \frac{1}{2} \lim_{t \rightarrow \infty} \left( -\frac{1}{e^{t^2}} + \frac{1}{e} \right) \\ &= \frac{1}{2} \left( 0 + \frac{1}{e} \right) = \frac{1}{2e}\end{aligned}$$

Since the improper integral converges, the series converges by the Integral Test.

(Note: the value of the series is not  $\frac{1}{2e}$ )

example:  $\sum_{n=1}^{\infty} \frac{\ln n}{n}$  converges or diverges?

Integral Test: Let  $f(x) = \frac{\ln x}{x}$ .

$f$  is continuous on  $[1, \infty)$  ✓

$f$  is positive on  $[1, \infty)$  since ✓  
 $\ln x \geq 0$  for  $x \geq 1$  and  $x > 0$  for  $x \geq 1$

$f$  decreasing?

$$f'(x) = \frac{(\frac{1}{x})x - 1 \cdot \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

denom. is always positive so  $f'(x) < 0$   
 when  $1 - \ln x < 0$   
 $1 < \ln x$   
 $e < x$

So  $f$  is decreasing on  $(e, \infty)$  but  
 not on  $(1, \infty)$

It turns out that decreasing after some point is good enough.

To see why, consider the series  $\sum_{n=3}^{\infty} \frac{\ln n}{n}$ .

Then  $f(x) = \frac{\ln x}{x}$  is continuous, positive and decreasing on  $[3, \infty)$  (since  $e < 3$ ).

We'll apply the Int. Test to  $\sum_{n=3}^{\infty} \frac{\ln n}{n}$   
 then argue that  $\sum_{n=1}^{\infty} \frac{\ln n}{n}$  has the same behavior.

$$\begin{aligned}\int_3^{\infty} \frac{\ln x}{x} dx &= \lim_{t \rightarrow \infty} \int_3^t \frac{\ln x}{x} dx & u &= \ln x \\ & & du &= \frac{1}{x} dx \\ &= \lim_{t \rightarrow \infty} \int_{\ln 3}^{\ln t} u du & x=3 &\Rightarrow u=\ln 3 \\ & & x=t &\Rightarrow u=\ln t \\ &= \lim_{t \rightarrow \infty} \left( \frac{1}{2} u^2 \right) \Big|_{\ln 3}^{\ln t} \\ &= \lim_{t \rightarrow \infty} \left( \frac{1}{2} (\ln t)^2 - \frac{1}{2} (\ln 3)^2 \right) \\ &= \infty \quad \text{diverges}\end{aligned}$$

$\therefore \sum_{n=3}^{\infty} \frac{\ln n}{n}$  diverges by the Integral Test.

$$\text{Since } \sum_{n=1}^{\infty} \frac{\ln n}{n} = \ln(1) + \frac{\ln(2)}{2} + \underbrace{\sum_{n=3}^{\infty} \frac{\ln n}{n}}_{\text{diverges}}$$

the series  $\sum_{n=1}^{\infty} \frac{\ln n}{n}$  also diverges.