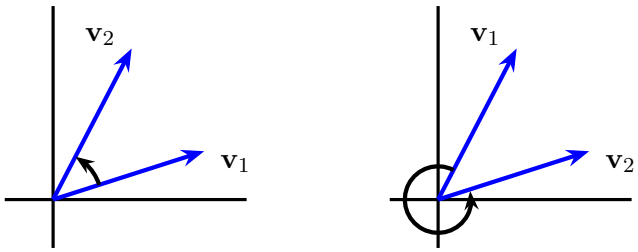
The determinant of a 2×2 matrix.

An ordered pair of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is called **positively oriented** if the angle from $\mathbf{v}_1$ and $\mathbf{v}_2$ measured in the counterclockwise direction is less than 180°; we say the pair is **negatively oriented** if it is more than 180°.



positively oriented negatively oriented

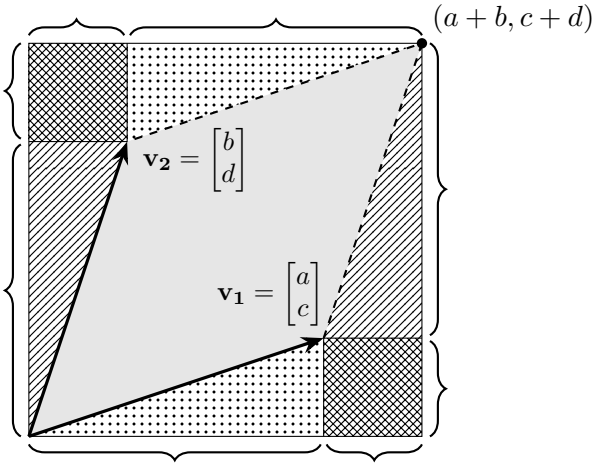
Definition (from ULA): Suppose a 2×2 matrix A has columns $\mathbf{v}_1$ and $\mathbf{v}_2$. If the pair of vectors is positively oriented, then the **determinant** of A, denoted $\det(A)$, is the area of the parallelogram formed by $\mathbf{v}_1$ and $\mathbf{v}_2$. If the pair is negatively oriented, then $\det(A)$ is -1 times the area of the parallelogram.

Ex: $\det\left(\begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}\right)$

Ex: $\det\left(\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\right)$

We also have the formula $\boxed{\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.}$

The connection between the two definitions is illustrated in this picture.



The determinant of a 3×3 matrix using Cofactor Expansion.

Expanding along the first row:

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}\right) = a\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ \cancel{d} & e & f \\ \cancel{g} & h & k \end{vmatrix} - b\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & \cancel{e} & f \\ g & \cancel{h} & k \end{vmatrix} + c\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & e & \cancel{f} \\ g & h & \cancel{k} \end{vmatrix} = a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ek - fh) - b(dk - fg) + c(dh - eg)$$

- The first term is the first entry in Row 1 times the determinant of the matrix you get by deleting the row and column containing that entry.
- The terms alternate sign, so the second term is $-b$ times the determinant of the matrix you get by deleting the 1st row and 2nd column.
- The third term is c times the determinant of the matrix you get by deleting the 1st row and 3rd column.

Computing $n \times n$ Determinants using Cofactor Expansion.

You can evaluate a determinant by expanding along any row or column (called the cofactor expansion along that row or column).

Definitions. Let A be an $n \times n$ matrix.

(a) The **(i, j)-minor** of A, denoted A_{ij}, is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith row and jth column.

(b) The **(i, j)-cofactor** of A, denoted C_{ij}, is given by the formula

$$C_{ij} = (-1)^{i+j} \det(A_{ij}).$$

The sign of the cofactor C_{ij} depends on the position of the entry a_{ij} in the matrix. (Here a_{ij} is the entry in the ith row and jth column.)

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \\ + & - & + & - & + & \\ - & + & - & + & - & \\ \vdots & & & & & \ddots \end{bmatrix}$$

Fact: $\det(AB) = \det(A)\det(B)$

From ULA Section 3.4:

> *Rather than simply thinking of the determinant as the area of a parallelogram, we may also think of it as a factor by which areas are scaled under the matrix transformation defined by the matrix. Applying the matrix transformation defined by B will scale area by $\det(B)$. If we then compose B with the matrix transformation defined by A, area will scale a second time by the factor $\det(A)$. The net effect is that the matrix transformation defined by AB scales area by $\det(A)\det(B)$ so that $\det(AB) = \det(A)\det(B)$.*