

# Partial Fraction Decomposition (PFD)

PFD is an algebra trick for pulling apart rational functions into simpler pieces.

Recall: A rational fun is  $\frac{p(x)}{q(x)}$   
where  $p, q$  are polynomials.

e.g.  $\frac{5x-4}{2x^2+x-1}$

example:  $\int \frac{5x-4}{2x^2+x-1} dx$

Can use PFD to write

$$\frac{5x-4}{2x^2+x-1} = \frac{3}{x+1} - \frac{1}{2x-1}$$

$$\int \frac{3}{x+1} - \frac{1}{2x-1} dx$$

$$= 3 \int \frac{1}{x+1} dx - \int \frac{1}{2x-1} dx$$

$$= 3 \ln|x+1| - \int \frac{1}{u} \cdot \frac{1}{2} du \quad \boxed{\begin{array}{l} u = 2x+1 \\ du = 2dx \end{array}}$$

$$= 3 \ln|x+1| - \frac{1}{2} \ln|u| + C$$

$$= 3 \ln|x+1| - \frac{1}{2} \ln|2x+1| + C$$

How to break apart the rational fun? <sup>guess</sup>

$$\frac{5x-4}{2x^2+x-1} = \frac{5x-4}{(2x-1)(x+1)} = \frac{A}{2x-1} + \frac{B}{x+1}$$

$$\cancel{(2x-1)}\cancel{(x+1)} \left( \frac{5x-4}{\cancel{(2x-1)}\cancel{(x+1)}} \right) = \left( \frac{A}{2x-1} + \frac{B}{x+1} \right) \cancel{(2x-1)}\cancel{(x+1)}$$

$$5x-4 = \frac{A\cancel{(2x-1)}(x+1)}{\cancel{(2x-1)}} + \frac{B(2x-1)\cancel{(x+1)}}{\cancel{(x+1)}}$$

don't  
touch  
this

$$\begin{cases} 5x-4 = A(x+1) + B(2x-1) \end{cases}$$

$$'' = Ax + A + 2Bx - B$$

$$5x-4 = (A+2B)x + (A-B)$$

$$A+2B=5$$

$$- ) A-B=-4$$

$$3B=9$$

$$\boxed{B=3}$$

$$\Rightarrow A=3-4=-1$$

$$\boxed{A=-1}$$

$$\int \frac{-1}{2x-1} + \frac{3}{x+1} dx$$

★ Important! For PFD to work, NEED degree of numerator strictly less than degree of denominator.

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There are some integrals that come up a lot in PFD problems:

$$\textcircled{1} \int \frac{1}{x+a} dx = \ln|x+a| + C$$

$$\begin{aligned} \textcircled{2} \int \frac{1}{(x+a)^2} dx &= \int \frac{1}{u^2} du \\ &= \int u^{-2} du \\ &= -u^{-1} + C \\ &= -\frac{1}{x+a} + C \end{aligned}$$

$u = x+a$   
 $du = dx$

$$\textcircled{3} \int \frac{1}{x^2+a^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

$$\left( \int \frac{1}{x^2+1} dx = \arctan(x) + C \right)$$

ex:  $\int \frac{x^2+4}{x(3x-2)(x+2)} dx$

deg(num) = 2  
deg(denom) = 3  
✓

PFD:

$$\overset{x}{(3x-2)} \overset{(x+2)}{\left( \frac{x^2+4}{x(3x-2)(x+2)} \right)} = \left( \frac{A}{x} + \frac{B}{3x-2} + \frac{C}{x+2} \right)$$

-  $x(3x-2)(x+2)$

$$x^2+4 = \underline{A(3x-2)(x+2)} + \underline{Bx(x+2)} + Cx(3x-2)$$

Trick: plug in roots of the denominator  
to  $x$  (here  $x=0, -2, 2/3$ )

$$x=0: \quad 4 = A(-2)(2) + 0 + 0 \Rightarrow -4A=4$$

$$\boxed{A=-1}$$

$$x=-2: \quad 8 = 0 + 0 + C(-2)(-8)$$

$$\Rightarrow \boxed{C = \frac{1}{2}}$$

$$x = \frac{2}{3}: \quad \frac{4}{9} + 4 = 0 + B \cdot \frac{2}{3} \left( \frac{2}{3} + 2 \right) + 0$$

$$\frac{40}{9} = B \cdot \frac{2}{3} \left( \frac{8}{3} \right)$$

$$B = \frac{40}{16} = \frac{10}{4} = \frac{5}{2}$$

$$\boxed{B = \frac{5}{2}}$$

$$\frac{x^2+4}{x(3x-2)(x+2)} = \frac{A}{x} + \frac{B}{3x-2} + \frac{C}{x+2}$$

$$= \frac{-1}{x} + \frac{(5/2)}{3x-2} + \frac{(1/2)}{x+2}$$

$$\int -\frac{1}{x} + \frac{(5/2)}{3x-2} + \frac{(1/2)}{x+2} dx$$

$$= -\int \frac{1}{x} dx + \frac{5}{2} \int \frac{1}{3x-2} dx + \frac{1}{2} \int \frac{1}{x+2} dx$$

$$= -\ln|x| + \frac{5}{2} \cdot \frac{1}{3} \ln|3x-2| + \frac{1}{2} \ln|x+2| + C$$

Choosing the correct form of your PFD:  
(ONLY depends on the denominator)

- if you have all distinct linear (degree 1) factors, in PFD you have each factor with a constant coefficient as a separate term

$$\frac{1}{x(x+1)(x+2)(x+3)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} + \frac{D}{x+3}$$

- repeated linear factor:

$$\frac{1}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$$

$$\frac{1}{(x-2)^3} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}$$

$$\frac{1}{(x+1)(x-2)^2} = \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

- irreducible quadratic factor

$$\frac{1}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$x^2+4$  is  
irreducible  
 $x^2+4=0$   
 $x^2=-4$   
no solns

- repeated irreducible quadratic factor

$$\frac{1}{x(x^2+4)^2} = \frac{1}{x} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$


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Pick correct form of the PFD

$$\textcircled{1} \quad \frac{1}{(x-1)^2(x+2)^3}$$

$$= \left( \frac{A}{x-1} + \frac{B}{(x-1)^2} \right) + \left( \frac{C}{x+2} + \frac{D}{(x+2)^2} + \frac{E}{(x+2)^3} \right)$$

$$\textcircled{2} \quad \frac{1}{x^4 - x^2} = \frac{1}{x^2(x^2-1)} = \frac{1}{x^2(x+1)(x-1)}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{x-1}$$

$$\textcircled{3} \quad \frac{1}{x^4 + x^2} = \frac{1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

↑ repeated linear      ↑ Irreducible quad.

$$\textcircled{4} \quad \frac{1}{x^2+2x+2} = \frac{Ax+B}{x^2+2x+2}$$

$$A=0, B=1$$

irreducible