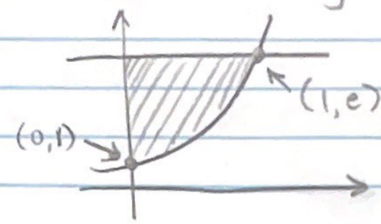
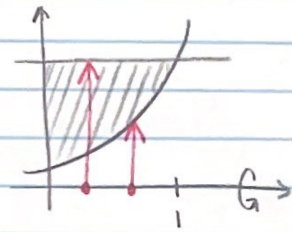
Method of Disks / Washers vs. Method of Shells

Consider the region bounded by $y = e^x$, $y = e$ and the y-axis.



Compute the volume of the solid of revolution obtained by rotating the region about the given line.

① rotated about the x-axis.



slice $\perp$ to the x-axis
$\Rightarrow$ use disks / washers
(integrate in x)

R = outer radius = e
r = inner radius = e^x

$$\Rightarrow A(x) = \pi(R^2 - r^2) = \pi(e^2 - (e^x)^2)$$

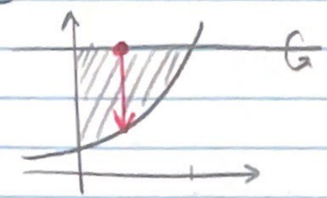
$$V = \int_0^1 \pi(e^2 - e^{2x})\,dx = \pi \int_0^1 e^2 - e^{2x}\,dx$$

bounds in terms of x so 0 to 1

$$= \pi\left(e^2 x - \tfrac{1}{2}e^{2x}\right)\Big|_0^1$$
$$= \pi\left(\left(e^2 - \tfrac{1}{2}e^2\right) - \left(0 - \tfrac{1}{2}e^0\right)\right)$$
$$= \pi\left(\tfrac{1}{2}e^2 + \tfrac{1}{2}\right)$$

Note: you can do this problem w/ shells but you would instead slice horizontally (parallel to the x-axis) and integrate in y. Means you need to convert $y = e^x$ to a function of y: $x = \ln(y)$

(2) rotated about $y=e$



again use disks/washers

radius $= e - e^x$

$A(x) = \pi r^2 = \pi(e-e^x)^2$, bounds in terms of x

$$V = \int_0^1 \pi(e-e^x)^2\,dx = \int_0^1 \pi\left(e^2 - 2e(e^x) + e^{2x}\right)dx$$

$$= \pi\left(\int_0^1 e^2\,dx - 2e\int_0^1 e^x\,dx + \int_0^1 e^{2x}dx\right)$$

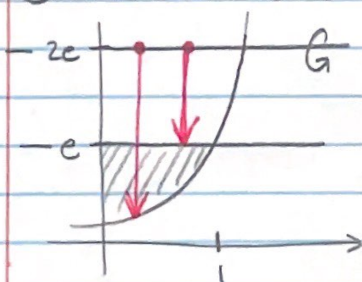
$$= \pi\left(e^2(1-0) - 2e(e^x)\Big|_0^1 + \left(\tfrac{1}{2}e^{2x}\right)\Big|_0^1\right)$$

$$= \pi\left(e^2 - 2e(e-1) + \left(\tfrac{1}{2}e^2 - \tfrac{1}{2}\right)\right)$$

$$= \pi\left(e^2 - 2e^2 + 2e + \tfrac{1}{2}e^2 - \tfrac{1}{2}\right)$$

$$= \pi\left(-\tfrac{1}{2}e^2 + 2e - \tfrac{1}{2}\right)$$

(3) rotated about $y=2e$



use disks/washers

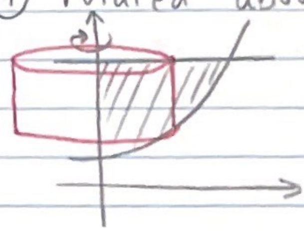
$R =$ outer radius $= 2e - e^x$
$r =$ inner radius $= 2e - e = e$

$A(x) = \pi\left((2e-e^x)^2 - (e)^2\right)$

$$V = \int_0^1 \pi\left((2e-e^x)^2 - e^2\right)dx = \pi\int_0^1 4e^2 - 4e(e^x) + e^{2x}dx$$

$$= \pi\left(4e^2 - 4e^2 + 4e + \tfrac{1}{2}e^2 - \tfrac{1}{2}\right) = \pi\left(\tfrac{1}{2}e^2 + 4e - \tfrac{1}{2}\right)$$

(4) rotated about the y-axis



slice parallel to axis of revolution
$\Rightarrow$ use shells (integrate in x)

height $= e - e^x$

$2\pi(\text{radius}) = 2\pi x$

$$V = \int_0^1 2\pi x(e - e^x)\,dx = 2\pi \int_0^1 xe - xe^x\,dx$$

$$= 2\pi\left(\int_0^1 ex\,dx - \int_0^1 xe^x\,dx\right)$$

$$= 2\pi\left(e\left(\tfrac{1}{2}x^2\right)\Big|_0^1 - \int_0^1 xe^x\,dx\right)$$

$$= 2\pi\left(\tfrac{e}{2}(1-0) - \int_0^1 xe^x\,dx\right)$$

IBP: $u = x \quad dv = e^x dx$
$du = dx \quad v = e^x$

$$= 2\pi\left(\tfrac{e}{2} - \left(xe^x\Big|_0^1 - \int_0^1 e^x\,dx\right)\right)$$

$$= 2\pi\left(\tfrac{e}{2} - \left((1 \cdot e^1 - 0) - e^x\Big|_0^1\right)\right)$$

$$= 2\pi\left(\tfrac{e}{2} - e + (e^1 - e^0)\right)$$

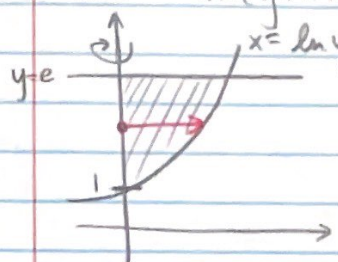
$$= 2\pi\left(-\tfrac{e}{2} + e - 1\right)$$

$$= 2\pi\left(e/2 - 1\right)$$

$$= \pi e - 2\pi$$

④ continued

OR integrate in y and use disks / washers



convert $y=e^x$ to $x=\ln y$

radius $= \ln y$

bounds in terms of y are 1 to e.

$$A(y) = \pi(\text{radius})^2 = \pi(\ln y)^2$$

$$V = \int_1^e \pi(\ln y)^2\,dy = \pi\int_1^e (\ln y)^2\,dy$$

$$\boxed{\begin{array}{ll} \text{IBP: } u=(\ln y)^2 & dv=dy \\ du = 2\ln y\cdot\frac{1}{y}\,dy & v=y \end{array}}$$

$$= \pi\left(\left((\ln y)^2 y\right)\Big|_1^e - \int_1^e \cancel{y}\cdot 2\ln y\cdot\frac{1}{\cancel{y}}\,dy\right)$$

$$= \pi\left(\left((\ln y)^2 y\right)\Big|_1^e - 2\int_1^e \ln y\,dy\right)$$

$$= \pi\left(\left((\ln(e))^2 e - (\ln(1))^2\right) - 2\int_1^e \ln y\,dy\right)$$

$$\boxed{\begin{array}{ll} \text{IBP: } u=\ln y & dv=dy \\ du = \frac{1}{y}\,dy & v=y \end{array}}$$

$$= \pi\left((e-0) - 2\left(y\ln y\Big|_1^e - \int_1^e y\cdot\frac{1}{y}\,dy\right)\right)$$

$$= \pi\left(e - 2\left(e\ln(e) - \cancel{1\cdot\ln(1)}^{\,0} - \int_1^e 1\,dy\right)\right)$$

$$= \pi(e - 2(e - (e-1)))$$

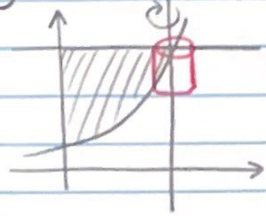
$$= \pi(e - 2e + 2(e-1))$$

$$= \pi(-e + 2e - 2)$$

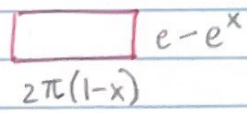
$$= \pi(e-2)$$

same answer!

5 rotated about $x=1$



slice parallel to $x=1 \Rightarrow$ shells



$$V=\int_0^1 2\pi(1-x)(e-e^x)\,dx = 2\pi\int_0^1 (1-x)(e-e^x)\,dx$$

$$= 2\pi \int_0^1 e - ex - e^x + xe^x\,dx$$

$$= 2\pi\left(\left(ex - e\left(\tfrac{1}{2}x^2\right) - e^x\right)\Big|_0^1 + \int_0^1 xe^x\,dx\right)$$

IBP: $u=x$ $\quad dv = e^x dx$
$du = dx$ $\quad v = e^x$

$$= 2\pi\left(\left(\cancel{e} - \tfrac{1}{2} - \cancel{e}\right) - (0 - 0 - e^0) + \left(xe^x\Big|_0^1 - \int_0^1 e^x\,dx\right)\right)$$

$$= 2\pi\left(-\tfrac{1}{2} + 1 + (e - 0) - e^x\Big|_0^1\right)$$

$$= 2\pi\left(\tfrac{1}{2} + \cancel{e} - (\cancel{e} - 1)\right)$$

$$= 2\pi\left(\tfrac{1}{2} + 1\right)$$

$$= 3\pi$$