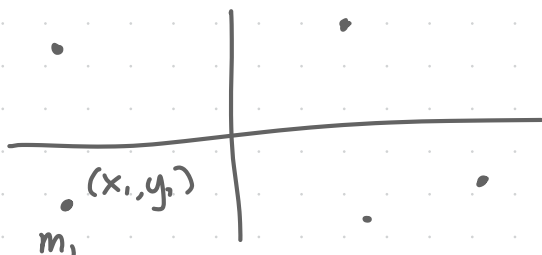


b.6- Center of Mass

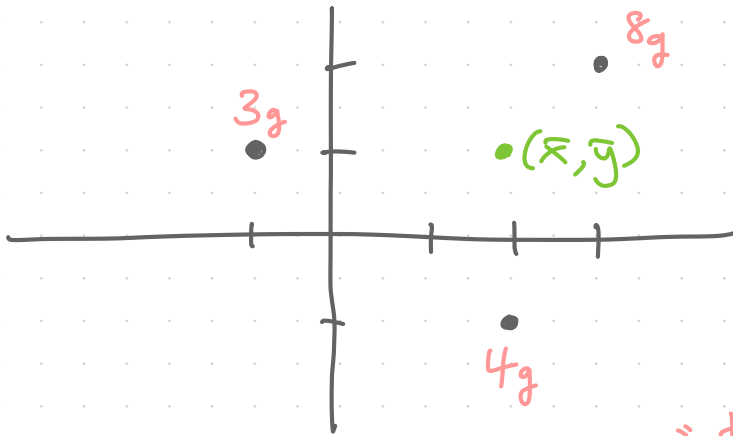
Point masses $m_1, m_2, \dots, m_n$ at positions $(x_1, y_1), \dots, (x_n, y_n)$



$$\bar{x} = \frac{\text{y-moment}}{\text{total mass}} = \frac{M_y}{m}$$

$$\bar{y} = \frac{\text{x-moment}}{\text{total mass}} = \frac{M_x}{m}$$

example: Find the CoM of a system of point masses 3, 4, and 8g at $(-1, 1)$, $(2, -1)$ and $(3, 2)$, respectively.



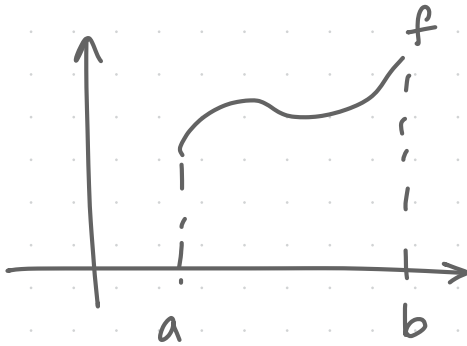
← mass ← "dist." to y-axis (i.e. the x-coord.)

$$\bar{x} = \frac{M_y}{m} = \frac{(3)(-1) + (4)(2) + 8(3)}{3 + 4 + 8} = \frac{29}{15}$$

$$\bar{y} = \frac{M_x}{m} = \frac{(3)(1) + (4)(-1) + (8)(2)}{3 + 4 + 8} = 1$$

CoM is $(\bar{x}, \bar{y})$

C.o.M Formulas



f on $[a, b]$
describes a region
in the plane

region / continuum $\rightarrow$ centroid
(density automatically
being $\rho=1$)

lamina $\xrightarrow{\hspace{2cm}}$ center of mass
(has density ρ)
 $\uparrow$
needed

$$\bar{x} = \frac{M_y}{m} = \frac{\int_a^b \cancel{\rho} \cdot x f(x) dx}{\int_a^b \cancel{\rho} f(x) dx} = \frac{1}{A} \int_a^b x f(x) dx$$

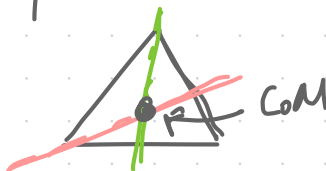
where A is area under f

$$\begin{aligned} \bar{y} &= \frac{M_x}{m} = \frac{\int_a^b \cancel{\rho} \cdot \frac{1}{2} f(x)^2 dx}{\int_a^b \cancel{\rho} f(x) dx} \\ &= \frac{1}{A} \int_a^b \frac{1}{2} f(x)^2 dx \end{aligned}$$

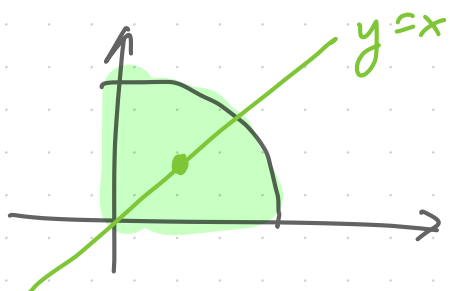
Using symmetry

Fact: If a region (with uniform density) is symmetric about a line l , then the COM will be on l .

ex: an equilateral triangle

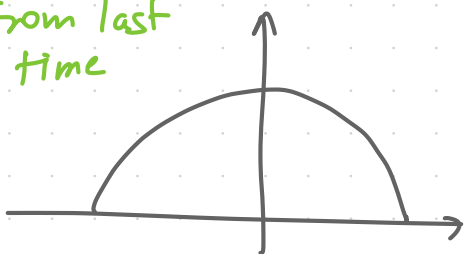


ex: a quarter circle



symmetry tells you
that $\bar{x} = \bar{y}$

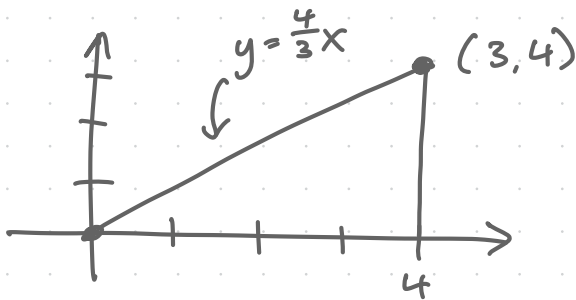
ex: from last
time



symmetry tells
us that $\bar{x} = 0$

by symmetry, $\bar{x} = 0$

ex: Find M_y , M_x and CoM of lamina with density $\rho = 2$.



$$\left[\text{Area under } f(x) = \frac{4}{3}x \right] = \frac{1}{2} (4)(3) = 6$$

$$\text{total mass} = \rho \cdot A = 2 \cdot 6 = 12$$

$$\left(\bar{x} = \frac{M_y}{m} = \frac{\int \rho \cdot x f(x) dx}{\rho \cdot \underbrace{\int_a^b f(x) dx}_A} \right)$$

$$M_y = \int_a^b \rho \cdot x f(x) dx$$

$$\begin{aligned} &= \int_0^4 2 \cdot x \left(\frac{4}{3}x \right) dx = \frac{8}{3} \left(\frac{1}{3} x^3 \right) \Big|_0^4 \\ &= \frac{8}{9} \cdot 4^3 = \frac{512}{9} \end{aligned}$$

$$\text{So } \bar{x} = \frac{M_y}{m} = \frac{(512/9)}{12} = \frac{128}{27}$$

$$\text{And } M_x = \int_a^b \rho \cdot \frac{1}{2} (f(x))^2 dx$$

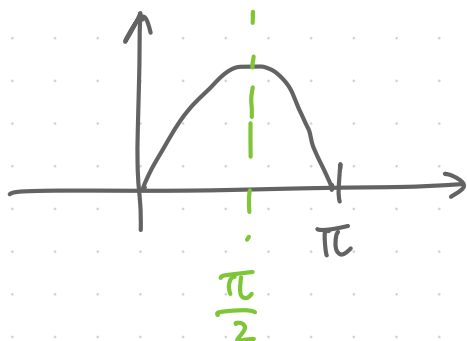
$$= \int_0^4 \cancel{2} \cdot \cancel{\frac{1}{2}} \left(\frac{4}{3}x\right)^2 dx$$

$$= \frac{16}{9} \int_0^4 x^2 dx$$

$$= \frac{16}{9} \left(\frac{1}{3} x^3\right) \Big|_0^4 = \frac{16}{27} \cdot 4^3 = \frac{1024}{27}$$

$$\bar{y} = \frac{M_x}{m} = \frac{(1024/27)}{12} = \frac{256}{81}$$

ex: Find the centroid of the region between $y = \sin x$ and x -axis for $0 \leq x \leq \pi$.



Assume $\rho = 1$.

By symmetry $\bar{x} = \pi/2$

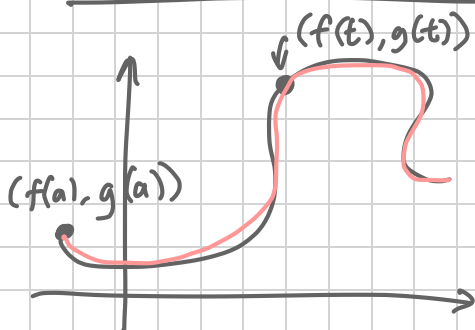
Need to compute: $\bar{y} = \frac{1}{A} \int_a^b \frac{1}{2} f(x)^2 dx$

$$A = \int_a^b f(x) dx = \int_0^{\pi} \sin x dx = 2$$

$$\bar{y} = \frac{1}{2} \int_0^{\pi} \frac{1}{2} \sin^2 x dx = \frac{\pi}{8}$$

Answer: $(\frac{\pi}{2}, \frac{\pi}{8})$

1.7 - Parametric Curves



fails horizontal
and vertical line
tests

$\Rightarrow$ neither a function
of x nor y

We can express this as a parametric curve

A parametric curve is described using parametric equations:

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \quad \text{where } a \leq t \leq b$$

f and g are function of the parameter t

For each t in $[a, b]$, get a point on the curve:

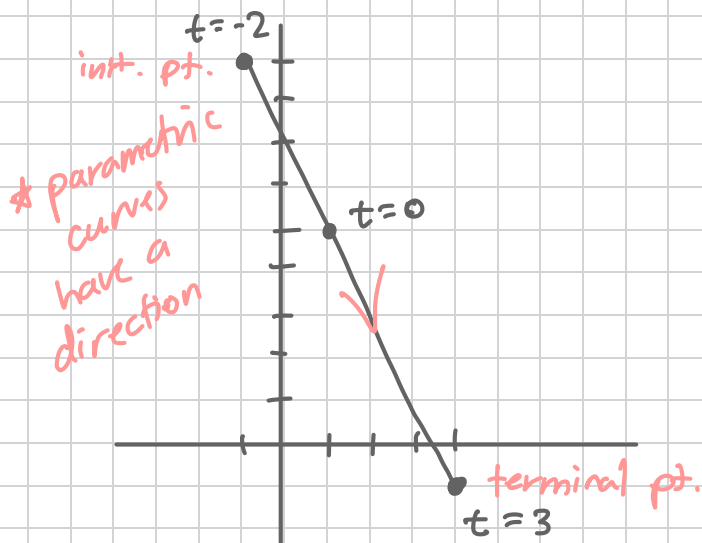
$$(x, y) = (f(t), g(t))$$

Alt.

Notation: $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$

example:

$$\begin{cases} x = 1+t \\ y = 5-2t \end{cases}$$
$$-2 \leq t \leq 3$$



t	x	y
-2	-1	9
0	1	5
3	4	-1

plotting points $\Rightarrow$ looks like a line

to know for sure, can eliminate the parameter

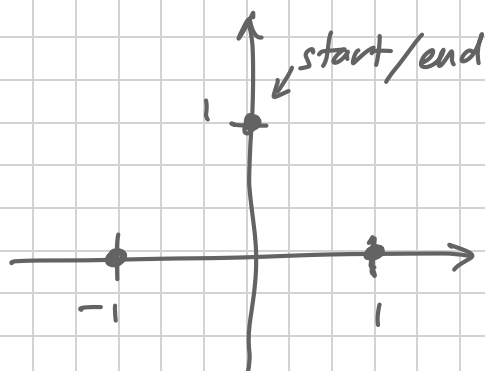
$$x = 1+t \quad \Rightarrow \quad t = x-1$$
$$y = 5-2t$$

$$y = 5-2(x-1)$$

$$y = 7-2x$$

ex: $\begin{cases} x = \sin t \\ y = 1 - \sin^2 t \end{cases}$ ★
 $0 \leq t \leq 2\pi$

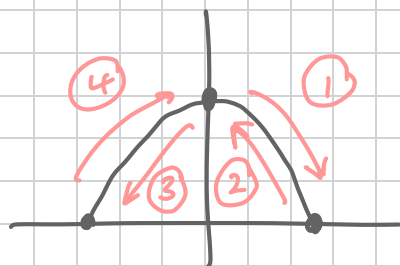
t	x	y
0	0	1
$\pi/2$	1	0
π	0	1
$3\pi/2$	-1	0
2π	0	1



strategy: eliminate the parameter to figure out the shape

$$x = \sin t \quad \& \quad y = 1 - (\sin t)^2$$

$$y = 1 - x^2$$



Note: a parametric curve can trace over itself

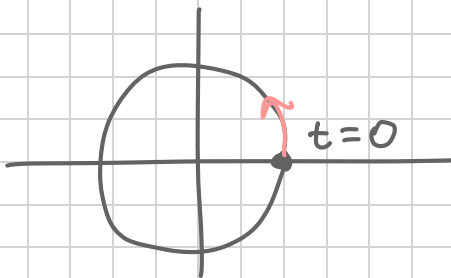
example:
$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$$
$$0 \leq t \leq 2\pi$$

unit circle

Trick for eliminating the parameter:

$$x^2 + y^2 = (\cos t)^2 + (\sin t)^2 = 1$$

$$x^2 + y^2 = 1$$



this parametrization
start at $(1, 0)$ and
goes counter clockwise

Variations of Previous Ex:

$$\begin{cases} x = \cos(2t) \\ y = \sin(2t) \end{cases}$$
$$0 \leq t \leq 2\pi$$

$$\begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases}$$
$$0 \leq t \leq 2\pi$$

$$\begin{cases} x = \sin t \\ y = \cos t \end{cases}$$
$$0 \leq t \leq \pi$$

$$\begin{cases} x = \cos t + 1 \\ y = \sin t - 2 \end{cases}$$
$$-\pi \leq t \leq \pi$$

shape, starting & end point, direction