The Octonionic Projective Plane

Inácio Rabelo

20/03/2024

ICMC USP SÃO CARLOS

Instituto de Matemáticas Unidad Cuernavaca

Contents

Basic Definition

Definition

A real normed division algebra $\mathbb{A}$ is a real vector space equipped with:

1. *A multiplication map with unity:*

$$m : \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{A}$$
$$m(1, a) = m(a, 1) = a \; \forall \; a \in \mathbb{A}$$

2. *The algebra has no zero divisors:*

$$ab = 0 \Rightarrow \; a = 0 \; or \; b = 0.$$

3. *A norm $\|.\|$.*

Complex and Quaternions

- The complex numbers $\mathbb{C}$ form a division algebra of dimension 2.
- The quaternions $\mathbb{K}$ form a non-commutative division algebra of dimension 4.

Complex and Quaternions

- The complex numbers $\mathbb{C}$ form a division algebra of dimension 2.
- The quaternions $\mathbb{K}$ form a non-commutative division algebra of dimension 4.
- The both are associative and constructed by adding imaginary units.

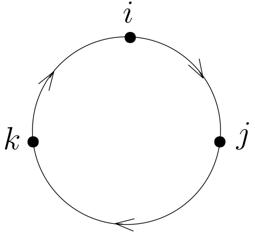


Figure: Quaternionic Multiplication

Octonions

- Octonions form a **non-associative** normed division algebra of dimension 8.

- Part of a general construction called Cayley-Dickson process.

Octonions

- Octonions form a **non-associative** normed division algebra of dimension 8.

- Part of a general construction called Cayley-Dickson process.

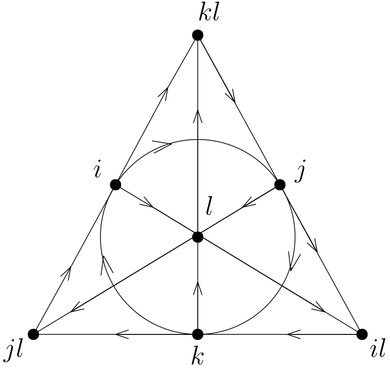


Figure: The Fano plane and the octonionic multiplication

Alternativity

Definition

An algebra $\mathbb{A}$ is alternative if $(aa)b = a(ab)$ and $(ba)a = b(aa)$ for all $a, b \in \mathbb{A}$.

Theorem (Artin)

An algebra is alternative if and only if the subalgebra generated by any two elements is associative.

Alternativity

Definition

An algebra $\mathbb{A}$ *is alternative if* $(aa)b = a(ab)$ *and* $(ba)a = b(aa)$ *for all* $a, b \in \mathbb{A}$*.*

Theorem (Artin)

An algebra is alternative if and only if the subalgebra generated by any two elements is associative.

Remark

The subalgebra $\mathbb{B} \subset \mathbb{A}$ *generated by two elements* $a, b \in \mathbb{A}$ *is the set generated by* $\{1, a, b\}$*.*

- $\mathbb{O}$ is alternative.
- In particular, $\mathbb{R} \subset \mathbb{B}$
- If $c \in \mathbb{B}$, then $c^{-1}, \bar{c} \in \mathbb{B}$.

Why are they important?

Theorem (Hurwitz)

The algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{K}$, $\mathbb{O}$ are, up to isomorphism, the only normed division algebras.

Theorem (Zorn)

The algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{K}$, $\mathbb{O}$ are, up to isomorphism, the only alternative division algebras.

Why are they important?

Theorem (Hurwitz)

The algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{K}$, $\mathbb{O}$ are, up to isomorphism, the only normed division algebras.

Theorem (Zorn)

The algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{K}$, $\mathbb{O}$ are, up to isomorphism, the only alternative division algebras.

Why are they important?

Theorem (Hurwitz)

The algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{K}$, $\mathbb{O}$ are, up to isomorphism, the only normed division algebras.

Theorem (Zorn)

The algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{K}$, $\mathbb{O}$ are, up to isomorphism, the only alternative division algebras.

Theorem (Bott-Milnor-Kervaire)

A division algebra has dimension 1,2,4, or 8.

Contents

A Big Picture

A quote

“The real numbers are the dependable breadwinner of the family, the complete ordered field we all rely on. The complex numbers are a slightly flashier but still respectable younger brother: not ordered, but algebraically complete. The quaternions, being noncommutative, are the eccentric cousin who is shunned at important family gatherings. But the octonions are the crazy old uncle nobody lets out of the attic: they are nonassociative.”

John Baez. The Octonions - Bull. Amer. Math. Soc. (2002)

A Big Picture

A quote

"The real numbers are the dependable breadwinner of the family, the complete ordered field we all rely on. The complex numbers are a slightly flashier but still respectable younger brother: not ordered, but algebraically complete. The quaternions, being noncommutative, are the eccentric cousin who is shunned at important family gatherings. But the octonions are the crazy old uncle nobody lets out of the attic: they are nonassociative."

John Baez. The Octonions - Bull. Amer. Math. Soc. (2002)

- Octonions and Octonionic Geometry describe models for exceptional structures.
- Applications in String Theory.
- Possible application in the "Theory of everything".

How the history started

- Lie algebras are vector spaces equipped with a bilinear product.
- They arise naturally from Lie Groups.

How the history started

- Lie algebras are vector spaces equipped with a bilinear product.
- They arise naturally from Lie Groups.
- There are a correspondence between Lie algebras and simply connected Lie groups.
- E. Cartan and W. Killing classified simple Lie algebras through their compact form.
- Exceptional structures emerged without any concrete model.

Classification of Simple Lie Algebras

Family	Complex Algebra	Complex Group	Compact Form	Real Group	Real Dimension
$A_n (n \geq 1)$	$\mathfrak{sl}(n+1;\mathbb{C})$	$\mathrm{SL}(n+1;\mathbb{C})$	$\mathfrak{su}(n+1)$	$\mathrm{SU}(n+1)$	$n(n+2)$
$B_n (n \geq 2)$	$\mathfrak{so}(2n+1;\mathbb{C})$	$\mathrm{SO}(2n+1;\mathbb{C})$	$\mathfrak{so}(2n+1)$	$\mathrm{Spin}(2n+1)$	$n(2n+1)$
$C_n (n \geq 3)$	$\mathfrak{sp}(n;\mathbb{C})$	$\mathrm{Sp}(n;\mathbb{C})$	$\mathfrak{sp}(n)$	$\mathrm{Sp}(n)$	$n(2n+1)$
$D_n (n \geq 4)$	$\mathfrak{so}(2n;\mathbb{C})$	$\mathrm{SO}(n;\mathbb{C})$	$\mathfrak{so}(n)$	$\mathrm{Spin}(n)$	$n(2n-1)$
$\mathbb{G}_2$	$\mathfrak{g}_2^{\mathbb{C}}$	${\mathrm{G}_2}^{\mathbb{C}}$	$\mathfrak{g}_2$	G_2	14
$\mathbb{F}_4$	$\mathfrak{f}_2^{\mathbb{C}}$	${\mathrm{F}_4}^{\mathbb{C}}$	$\mathfrak{f}_4$	F_4	52
$\mathbb{E}_6$	$\mathfrak{e}_6^{\mathbb{C}}$	$\mathrm{E}_6^{\mathbb{C}}$	$\mathfrak{e}_6$	E_6	78
$\mathbb{E}_7$	$\mathfrak{e}_7^{\mathbb{C}}$	$\mathrm{E}_7^{\mathbb{C}}$	$\mathfrak{e}_7$	E_7	133
$\mathbb{E}_8$	$\mathfrak{e}_8^{\mathbb{C}}$	$\mathrm{E}_8^{\mathbb{C}}$	$\mathfrak{e}_8$	E_8	248

Table: Classification of Complex Simple Lie Algebras

Classification of Simple Lie Algebras

Family	Complex Algebra	Complex Group	Compact Form	Real Group	Real Dimension
$A_n (n \geq 1)$	$\mathfrak{sl}(n+1;\mathbb{C})$	$\mathrm{SL}(n+1;\mathbb{C})$	$\mathfrak{su}(n+1)$	$\mathrm{SU}(n+1)$	$n(n+2)$
$B_n (n \geq 2)$	$\mathfrak{so}(2n+1;\mathbb{C})$	$\mathrm{SO}(2n+1;\mathbb{C})$	$\mathfrak{so}(2n+1)$	$\mathrm{Spin}(2n+1)$	$n(2n+1)$
$C_n (n \geq 3)$	$\mathfrak{sp}(n;\mathbb{C})$	$\mathrm{Sp}(n;\mathbb{C})$	$\mathfrak{sp}(n)$	$\mathrm{Sp}(n)$	$n(2n+1)$
$D_n (n \geq 4)$	$\mathfrak{so}(2n;\mathbb{C})$	$\mathrm{SO}(n;\mathbb{C})$	$\mathfrak{so}(n)$	$\mathrm{Spin}(n)$	$n(2n-1)$
$\mathbb{G}_2$	$\mathfrak{g}_2^{\mathbb{C}}$	${\mathrm{G}_2}^{\mathbb{C}}$	$\mathfrak{g}_2$	G_2	14
$\mathbb{F}_4$	$\mathfrak{f}_2^{\mathbb{C}}$	${\mathrm{F}_4}^{\mathbb{C}}$	$\mathfrak{f}_4$	F_4	52
$\mathbb{E}_6$	$\mathfrak{e}_6^{\mathbb{C}}$	$\mathrm{E}_6^{\mathbb{C}}$	$\mathfrak{e}_6$	E_6	78
$\mathbb{E}_7$	$\mathfrak{e}_7^{\mathbb{C}}$	$\mathrm{E}_7^{\mathbb{C}}$	$\mathfrak{e}_7$	E_7	133
$\mathbb{E}_8$	$\mathfrak{e}_8^{\mathbb{C}}$	$\mathrm{E}_8^{\mathbb{C}}$	$\mathfrak{e}_8$	E_8	248

Table: Classification of Complex Simple Lie Algebras

Contents

History

- Its origins remount to Renaissance in the context of Perspective Drawing.
- Based on the idea of two lines intersect at infinity.
- Non-existence of parallel lines.

History

- Its origins remount to Renaissance in the context of Perspective Drawing.
- Based on the idea of two lines intersect at infinity.
- Non-existence of parallel lines.

Figure: School of Atens - Raphael

Axioms

Definition

A projective plane $\mathbb{P}$ is a triple consisting of a set of points, a set of lines, and a relation whereby a point ***lies on*** *a line, also called incidence relation, satisfying the following axioms:*

Axioms

Definition

A projective plane $\mathbb{P}$ is a triple consisting of a set of points, a set of lines, and a relation whereby a point ***lies on*** *a line, also called incidence relation, satisfying the following axioms:*

1. *For any two distinct points, there is a unique line on which they both lie.*
2. *For any two distinct lines, there is a unique point which lies on both of them.*
3. *There exist four points, no three of which lie on the same line.*
4. *There exist four lines, no three of which have the same point lying on them.*

The Classical Cases

- Notation: $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{K}$.
- $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3) \in \mathbb{F}^3 - \{0\}$.

The Classical Cases

- Notation: $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{K}$.
- $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3) \in \mathbb{F}^3 - \{0\}$.
- Equivalence relation:

$$v \sim w \iff \exists\, \lambda \in \mathbb{F}^* \text{ such that } v_i = \lambda w_i \;\forall\; i = 1, 2, 3.$$

- In particular: $u_i = \lambda'(\lambda v_i) = (\lambda' \lambda) v_i$, where $\lambda, \lambda' \in \mathbb{F}$.

The Classical Cases

- Notation: $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{K}$.
- $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3) \in \mathbb{F}^3 - \{0\}$.
- Equivalence relation:

$$v \sim w \iff \exists\, \lambda \in \mathbb{F}^* \text{ such that } v_i = \lambda w_i \;\forall\, i = 1, 2, 3.$$

- In particular: $u_i = \lambda'(\lambda v_i) = (\lambda'\lambda)v_i$, where $\lambda, \lambda' \in \mathbb{F}$.

Definition

$$\mathbb{FP}^2 = \mathbb{F}^3 - \{0\}/\sim .$$

The Classical Cases

- Notation: $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{K}$.
- $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3) \in \mathbb{F}^3 - \{0\}$.
- Equivalence relation:

$$v \sim w \iff \exists\, \lambda \in \mathbb{F}^* \text{ such that } v_i = \lambda w_i \;\forall\; i = 1, 2, 3.$$

- In particular: $u_i = \lambda'(\lambda v_i) = (\lambda'\lambda)v_i$, where $\lambda, \lambda' \in \mathbb{F}$.

Definition

$$\mathbb{FP}^2 = \mathbb{F}^3 - \{0\}/\sim .$$

Remark

For the quaternionic case $\mathbb{K}^3$, we consider a right vector space structure.

Illustration

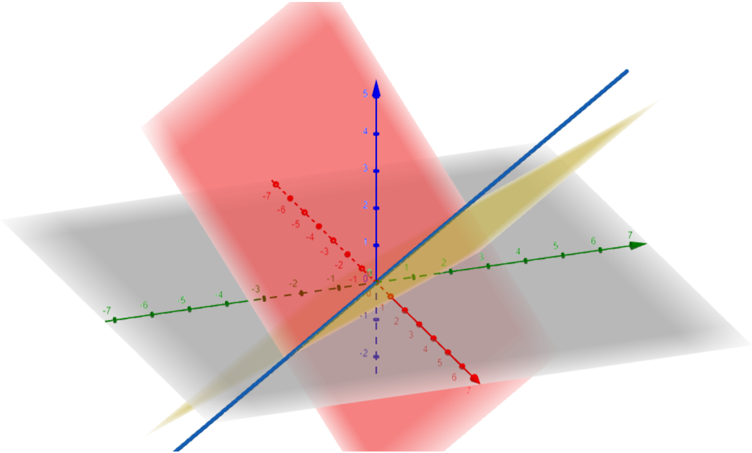


Figure: Planes intersecting according to a line in $\mathbb{R}^3$

Contents

Why this does not work for $\mathbb{O}$

- $\mathbb{O}^3$ is not a vector space: $(z\lambda_1)\lambda_2 \neq z(\lambda_1\lambda_2)$, $z \in \mathbb{O}^3, \lambda_i \in \mathbb{O}$.
- The previous relation $\sim$ does not define an equivalence relation.

Why this does not work for $\mathbb{O}$

- $\mathbb{O}^3$ is not a vector space: $(z\lambda_1)\lambda_2 \neq z(\lambda_1\lambda_2)$, $z \in \mathbb{O}^3, \lambda_i \in \mathbb{O}$.
- The previous relation $\sim$ does not define an equivalence relation.
- Ruth Moufang constructed projective planes **coordinatized** by non-associative algebras (1933).

Why this does not work for $\mathbb{O}$

- $\mathbb{O}^3$ is not a vector space: $(z\lambda_1)\lambda_2 \neq z(\lambda_1\lambda_2)$, $z \in \mathbb{O}^3, \lambda_i \in \mathbb{O}$.
- The previous relation $\sim$ does not define an equivalence relation.
- Ruth Moufang constructed projective planes **coordinatized** by non-associative algebras (1933).
- Characterizations due to H. Freudenthal, E. Borel, P. Jordan, and M. Hirsh.
- Jordan algebras: A substitute for the vector space structure.

What is a coordinatization?

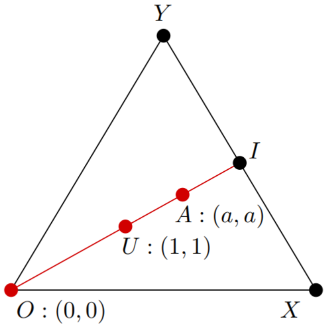
- Choose three lines: XY, YO, and OU.
- Their existence follows from the axioms.

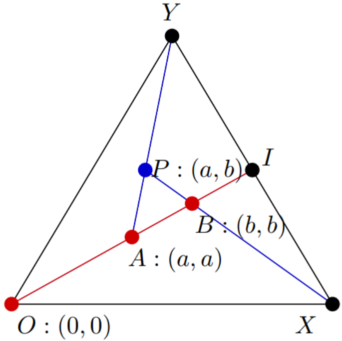
What is a coordinatization?

- Choose three lines: XY, YO, and OU.
- Their existence follows from the axioms.
- It is possible to coordinatize the plane by the line except a point.
- The geometry of the plane induces an algebraic structure on this line.

What is a coordinatization?

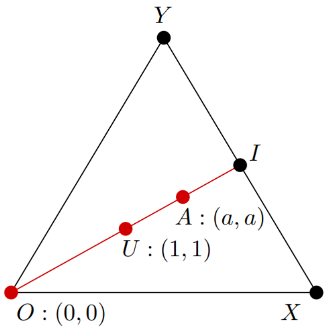
- Choose three lines: XY, YO, and OU.
- Their existence follows from the axioms.
- It is possible to coordinatize the plane by the line except a point.
- The geometry of the plane induces an algebraic structure on this line.

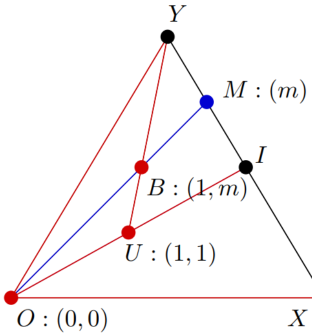




What is a coordinatization?

- Choose three lines: XY, YO, and OU.
- Their existence follows from the axioms.
- It is possible to coordinatize the plane by the line except a point.
- The geometry of the plane induces an algebraic structure on this line.





Matrix Model for the Classical Planes

- $\mathbf{H}_3(\mathbb{F})$: Hermitian matrices with entries in $\mathbb{F}$.

- Let L_v be the line generated by $v \in \mathbb{F}^3 - \{0\}$, with $\|v\| = 1$.

- Let A be the matrix which denotes the projection on L_v.

Matrix Model for the Classical Planes

- $\mathbf{H}_3(\mathbb{F})$: Hermitian matrices with entries in $\mathbb{F}$.
- Let L_v be the line generated by $v \in \mathbb{F}^3 - \{0\}$, with $\|v\| = 1$.
- Let A be the matrix which denotes the projection on L_v.
- $A \in \mathbf{H}_3(\mathbb{F})$, $A^2 = A$ and $\mathbf{tr}(A) = 1$.
- A is of the form

$$A = v\bar{v}^t = \begin{pmatrix} |v_1|^2 & v_1\bar{v}_2 & v_1\bar{v}_3 \\ v_2\bar{v}_1 & |v_2|^2 & v_2\bar{v}_3 \\ v_3\bar{v}_1 & v_3\bar{v}_2 & |v_3|^2 \end{pmatrix}.$$

Continuation

- Conversely, let $A \in \mathbf{H}_3(\mathbb{F})$ with $A^2 = A$.
- A is a projection on $W = \{w \in \mathbb{F}^3 : A(w) = w\}$.

Continuation

- Conversely, let $A \in \mathbf{H}_3(\mathbb{F})$ with $A^2 = A$.
- A is a projection on $W = \{w \in \mathbb{F}^3 : A(w) = w\}$.
- A is diagonalizable by an unitary matrix in $\mathbb{R}, \mathbb{C}$ or $\mathbb{K}$.
- It holds that $\mathbf{tr}(A) = \dim(W)$.
- If $\mathbf{tr}(A) = 1$, choose a unit vector $w \in W$. Then $A = w\bar{w}^t$.

Continuation

- Conversely, let $A \in \mathbf{H}_3(\mathbb{F})$ with $A^2 = A$.
- A is a projection on $W = \{w \in \mathbb{F}^3 : A(w) = w\}$.
- A is diagonalizable by an unitary matrix in $\mathbb{R}, \mathbb{C}$ or $\mathbb{K}$.
- It holds that $\mathbf{tr}(A) = \dim(W)$.
- If $\mathbf{tr}(A) = 1$, choose a unit vector $w \in W$. Then $A = w\bar{w}^t$.
- $\mathbb{FP}^2 = \{A \in \mathbf{H}_3(\mathbb{F}) : A^2 = A \text{ and } \mathbf{tr}(A) = 1\}$.

Jordan Algebras

Definition

A Jordan algebra $\mathbf{J}$ *is an algebra with a bilinear Jordan product denoted by* $X \circ Y$ *satisfying:*

1. $X \circ Y = Y \circ X$
2. $(X^2 \circ Y) \circ X = X^2 \circ (Y \circ X)$

for all $X, Y \in \mathbf{J}$.

Jordan Algebras

Definition

A Jordan algebra $\mathbf{J}$ *is an algebra with a bilinear Jordan product denoted by* $X \circ Y$ *satisfying:*

1. $X \circ Y = Y \circ X$
2. $(X^2 \circ Y) \circ X = X^2 \circ (Y \circ X)$

for all $X, Y \in \mathbf{J}$.

Theorem (Albert)

The set

$$\mathbf{H}_3(\mathbb{O}) = \{X \in \mathbf{M}_3(\mathbb{O}) : X^* = X\}$$

equipped with the product $X \circ Y = 1/2(XY + YX)$ *is an real Jordan algebra of dimension 27 which is not associative.*

Forms in $\mathbf{H}_3(\mathbb{O})$

Definition

In $\mathbf{H}_3(\mathbb{O})$ are defined:

- *A quadratic form by* $Q(X) = \frac{1}{2}\mathbf{tr}(X^2)$.
- *The correspondent bilinear form:* $(X, Y) = \frac{1}{2}\mathbf{tr}(X \circ Y)$.
- *The Freudenthal product by*

$$X * Y = X \circ Y - \frac{1}{4}(X\mathbf{tr}(Y) + Y\mathbf{tr}(X)) + \frac{1}{4}I_3(\mathbf{tr}(X)\mathbf{tr}(Y) - \mathbf{tr}(X \circ Y)).$$

The plane $\mathbb{OP}^2$

Definition

1. *The set of points (and lines) in* $\mathbb{OP}^2$ *is*

$$\mathbb{OP}^2 = \{X \in \mathbf{H}_3(\mathbb{O}) : X \neq 0,\ X * X = 0\}$$

up to multiplication by a real scalar.

2. *Equivalently,* $\mathbb{OP}^2 \simeq \{X \in \mathbf{H}_3(\mathbb{O}) : X^2 = X$ *and* $\mathbf{tr}(X) = 1\}$.

3. *A point* X *lies on a line* Y *if*

$$(X, Y) = \frac{1}{2}\mathbf{tr}(X \circ Y) = 0.$$

Characterization of Points

Theorem

Every point $X \in \mathbb{OP}^2$ has the following form

$$z\overline{z}^t = \begin{pmatrix} |z_1|^2 & z_1\overline{z_2} & z_1\overline{z_3} \\ z_2\overline{z_1} & |z_2|^2 & z_2\overline{z_3} \\ z_3\overline{z_1} & z_3\overline{z_2} & |z_3|^2 \end{pmatrix},$$

where $z = (z_1, z_2, z_3) \in \mathbb{O}^3$ lies on an associative subalgebra.

Octoninic Projective Line

Definition

1. *The set of points in* $\mathbb{OP}^1$ *is*

$$\mathbb{OP}^1 = \{X \in \mathbf{H}_2(\mathbb{O}) : X^2 = X \text{ and } \mathbf{tr}(X) = 1\}.$$

2. *All points lie on the (identity matrix) line* $\boldsymbol{I}_2$*.*

Octoninic Projective Line

Definition

1. *The set of points in* $\mathbb{OP}^1$ *is*

$$\mathbb{OP}^1 = \{X \in \mathbf{H}_2(\mathbb{O}) : X^2 = X \textit{ and } \mathbf{tr}(X) = 1\}.$$

2. *All points lie on the (identity matrix) line* $\boldsymbol{I}_2$.

Theorem

1. $\mathbb{RP}^1 \simeq \mathbb{S}^1$.
2. $\mathbb{CP}^1 \simeq \mathbb{S}^2$.
3. $\mathbb{KP}^1 \simeq \mathbb{S}^4$.
4. $\mathbb{OP}^1 \simeq \mathbb{S}^8$

The Exceptional Group F_4

Definition

A linear map $g \in \mathrm{GL}(\mathbf{H}_3(\mathbb{O}))$ *is an automorphism of the Jordan Algebra* $\mathbf{H}_3(\mathbb{O})$ *if*

$$g(A \circ B) = g(A) \circ g(B)$$

for all $A, B \in \mathbf{H}_3(\mathbb{O})$.

The Exceptional Group F_4

Definition

A linear map $g \in \mathrm{GL}(\mathbf{H}_3(\mathbb{O}))$ *is an automorphism of the Jordan Algebra* $\mathbf{H}_3(\mathbb{O})$ *if*

$$g(A \circ B) = g(A) \circ g(B)$$

for all $A, B \in \mathbf{H}_3(\mathbb{O})$.

Theorem (Chevalley-Shafer, 1950)

The compact simply connected real Lie group of type $\mathbb{F}_4$ *is the automorphism group of the Jordan algebra* $\mathbf{H}_3(\mathbb{O})$ *and it is denoted by* F_4.

Homogeneous Space

Theorem (Borel-Freudenthal,1950)
Let $E_1 \in \mathbb{OP}^2$ *be the diagonal matrix* $\mathbf{diag}(1,0,0)$.

- *The action of* F_4 *on* $\mathbb{OP}^2$ *is transitive.*
- *The isotropy group at* E_1 *is equal to* $\mathrm{Spin}(9)$.

Thus

$$\mathbb{OP}^2 \simeq \mathrm{F}_4 \,/\, \mathrm{Spin}(9)\,.$$

Contents

Non-existence of Higher-Dimensional Spaces

Theorem (Veblen-Young, 1910's)

Suppose $\mathbb{P}^n$ *is an arbitrary projective space with dimension* $n \geq 3$*. Then* $\mathbb{P}^n$ *is the projective space obtained from a left or right vector space* $\mathbb{D}^{n+1}$ *for some division ring* $\mathbb{D}$*.*

Non-existence of Higher-Dimensional Spaces

Theorem (Veblen-Young, 1910's)

Suppose $\mathbb{P}^n$ is an arbitrary projective space with dimension $n \geq 3$. Then $\mathbb{P}^n$ is the projective space obtained from a left or right vector space $\mathbb{D}^{n+1}$ for some division ring $\mathbb{D}$.

Corollary

There does not exist an octonionic projective space $\mathbb{O}\mathbb{P}^n$, with $n \geq 3$.

Non-existence of Higher-Dimensional Spaces

Theorem (Veblen-Young, 1910's)

Suppose $\mathbb{P}^n$ is an arbitrary projective space with dimension $n \geq 3$. Then $\mathbb{P}^n$ is the projective space obtained from a left or right vector space $\mathbb{D}^{n+1}$ for some division ring $\mathbb{D}$.

Corollary

There does not exist an octonionic projective space $\mathbb{OP}^n$, with $n \geq 3$.

Proof.

- Suppose the existence of a higher dimensional space over $\mathbb{O}$.

Non-existence of Higher-Dimensional Spaces

Theorem (Veblen-Young, 1910's)

Suppose $\mathbb{P}^n$ is an arbitrary projective space with dimension $n \geq 3$. Then $\mathbb{P}^n$ is the projective space obtained from a left or right vector space $\mathbb{D}^{n+1}$ for some division ring $\mathbb{D}$.

Corollary

There does not exist an octonionic projective space $\mathbb{O}\mathbb{P}^n$, with $n \geq 3$.

Proof.

- Suppose the existence of a higher dimensional space over $\mathbb{O}$.
- It would induces an associative structure on the space $\mathbb{O} \simeq \mathbb{R}^8$.
- This contradicts Zorn's Theorem.

□

Classical Jordan Algebras

- $\mathbf{H}_3(\mathbb{F}) = \{A \in \mathbf{M}_3(\mathbb{F}) : A^* = A\}$.
- Jordan product given by $X \circ Y = 1/2(XY + YX)$.
- Alternative projective description of $\mathbb{FP}^2$.

Classical Jordan Algebras

- $\mathbf{H}_3(\mathbb{F}) = \{A \in \mathbf{M}_3(\mathbb{F}) : A^* = A\}$.
- Jordan product given by $X \circ Y = 1/2(XY + YX)$.
- Alternative projective description of $\mathbb{FP}^2$.

Theorem

1. $\mathbf{Aut}_0(\mathbf{H}_3(\mathbb{R})) = \mathrm{SO}(3) = \mathbf{Iso}_0(\mathbb{RP}^2)$.
2. $\mathbf{Aut}_0(\mathbf{H}_3(\mathbb{C})) = \mathrm{PSU}(3) = \mathbf{Iso}_0(\mathbb{CP}^2)$.
3. $\mathbf{Aut}_0(\mathbf{H}_3(\mathbb{K})) = \mathrm{PSp}(3) = \mathbf{Iso}_0(\mathbb{KP}^2)$.

Classical Jordan Algebras

- $\mathbf{H}_3(\mathbb{F}) = \{A \in \mathbf{M}_3(\mathbb{F}) : A^* = A\}$.
- Jordan product given by $X \circ Y = 1/2(XY + YX)$.
- Alternative projective description of $\mathbb{FP}^2$.

Theorem

1. $\mathbf{Aut}_0(\mathbf{H}_3(\mathbb{R})) = \mathrm{SO}(3) = \mathbf{Iso}_0(\mathbb{RP}^2)$.
2. $\mathbf{Aut}_0(\mathbf{H}_3(\mathbb{C})) = \mathrm{PSU}(3) = \mathbf{Iso}_0(\mathbb{CP}^2)$.
3. $\mathbf{Aut}_0(\mathbf{H}_3(\mathbb{K})) = \mathrm{PSp}(3) = \mathbf{Iso}_0(\mathbb{KP}^2)$.

Conclusion

The Jordan algebras $\mathbf{H}_3(\mathbb{F})$ *also replace the vector space structure in the classical cases.*

Gracias!