Properties of the Dot Product: If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^n$ and $c \in \mathbb{R}$, then

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
- $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

Properties of the Norm: If $\mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^n$ and $c \in \mathbb{R}$, then

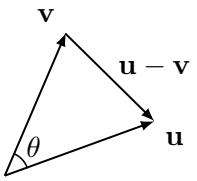
- $\|\mathbf{v}\| \geq 0$
- If $\|\mathbf{v}\| = 0$, then $\mathbf{v} = \mathbf{0}$ (the only vector with norm 0 is the zero vector)
- $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$
- The triangle inequality: $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$

Other facts/definitions:

- $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ and $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
- $\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$
- A **unit vector** is a vector of norm 1. If $\mathbf{v} \neq \mathbf{0}$, then $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector in the same direction as $\mathbf{v}$.
- We say $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$

Connection between the Dot Product and Angles between Vectors

Recall the Law of Cosines, which says that for a triangle with side lengths a, b and c, $\boxed{c^2 = a^2 + b^2 - 2ab\cos\theta}$ where θ is the angle opposite the side with length c. Consider two vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^2$ with angle θ between them.



Then the Law of Cosines says that $\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$. Rearranging terms and expanding, we get

$$\begin{aligned} 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2 \\ &= (u_1^2 + u_2^2) + (v_1^2 + v_2^2) - \big((u_1 - v_1)^2 + (u_2 - v_2)^2\big) \\ &= u_1^2 + u_2^2 + v_1^2 + v_2^2 - \big(u_1^2 - 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2\big) \\ &= 2u_1v_1 + 2u_2v_2 = 2(\mathbf{u} \cdot \mathbf{v}) \end{aligned}$$

Therefore, $\boxed{\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta.}$

In $\mathbb{R}^n$, we define the **angle between two vectors** $\mathbf{u}$ and $\mathbf{v}$ as the angle θ that satisfies $\boxed{\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}.}$

1. Consider the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$.

   (a) Find $\mathbf{u} \cdot \mathbf{u}$, $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \cdot \mathbf{w}$.

   (b) Use the dot products you found in part (a) to evaluate the following.

   i. $|\mathbf{u}|$

   ii. $(-5\mathbf{u}) \cdot \mathbf{v}$

   iii. $\mathbf{u} \cdot (-3\mathbf{v} + 2\mathbf{w})$

   iv. $\left| \frac{1}{|\mathbf{u}|} \mathbf{u} \right|$

   (c) For what value of k is $\mathbf{u}$ orthogonal to $k\mathbf{v} + 5\mathbf{w}$?

2. Suppose that $\mathbf{v}$ and $\mathbf{w}$ are vectors where $\mathbf{v} \cdot \mathbf{v} = 4$, $\mathbf{w} \cdot \mathbf{w} = 16$ and $\mathbf{v} \cdot \mathbf{w} = 4$.

   (a) What is $|\mathbf{v}|$?

   (b) What is the angle between $\mathbf{v}$ and $\mathbf{w}$?

3. If $\|\mathbf{v}\| = \mathbf{5}$ and $\|\mathbf{w}\| = \mathbf{3}$, what are the smallest and largest possible values of $\|\mathbf{v} - \mathbf{w}\|$? What are the smallest and largest possible values of $\mathbf{v} \cdot \mathbf{w}$?