**Taylor's Inequality**

The remainder of the Taylor series for $f(x)$ centered at $a$ is defined as

$$R_n(x) = f(x) - T_n(x),$$

where $T_n(x)$ denotes the $n$th degree Taylor polynomial (also centered at $a$). Taylor's Inequality says that if $\left|f^{(n+1)}(x)\right| \leq M$ for all $x$ satisfying $|x-a| \leq d$, then

$$\left|R_n(x)\right| \leq \frac{M}{(n+1)!}\left|x-a\right|^{n+1} \quad \text{for } x \text{ satisfying } |x-a| \leq d.$$

1. Use Taylor's Inequality to determine the number of terms of the Maclaurin series for $f(x) = \sin(x)$ needed to compute $\sin(3°)$ to within 0.000005.

2. (a) Find the 2nd degree Taylor polynomial, $T_2(x)$ for $f(x) = x^{-2}$ based at $a = 1$.
   (b) Use Taylor's Inequality to estimate the accuracy of the approximation using $T_2(x)$ to estimate $f(x)$ in the interval $[0.9, 1.1]$.

If you're using a Taylor polynomial to approximate $f$ at a particular value of $x$ instead of a range of $x$, the following formulation of Taylor's Inequality is sometimes useful.

**Alternate Statement of Taylor's Inequality**

If $\left|f^{(n+1)}(x)\right| \leq M$ for all $x$ between $a$ and $x_0$, then

$$\left|R_n(x_0)\right| \leq \frac{M}{(n+1)!}\left|x_0 - a\right|^{n+1}.$$

3. How large should $n$ be to guarantee that the approximation of $\ln(0.5)$ using an $n$th degree Taylor polynomial for $\ln(1+x)$ centered at $a = 0$ is within 0.0001?

4. Estimate the range of values of $x$ for which the approximation

$$\ln(x) \approx \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2$$

is accurate to within 0.01.