

1. Consider the two following lines: one with vector equation $\mathbf{r}_1(s) = \langle 4 - 2s, -2 + s, 1 + 3s \rangle$ and the other being the line through the point $(-4, 2, 17)$ in the direction $\mathbf{v} = \langle -2, 1, 5 \rangle$.

(a) Find the direction of the first line.

$$\langle -2, 1, 3 \rangle$$

(b) Find a vector equation for the second line, written in terms of the parameter t .

$$\begin{aligned}\vec{r}_2(t) &= \vec{r}_0 + t\vec{v} \\ &= \langle -4, 2, 17 \rangle + t\langle -2, 1, 5 \rangle \\ &= \langle -4 - 2t, 2 + t, 17 + 5t \rangle\end{aligned}$$

(c) Show that the two lines intersect by finding values of s and t that result in the same point. What is the point of intersection?

$$\begin{aligned}-4 - 2t &= 4 - 2s \\ 2 + t &= -2 + s \rightarrow s = t + 4 \\ 17 + 5t &= 1 + 3s \\ 5t - 3s &= -16 \\ 5t - 3(t + 4) &= -16 \\ 2t - 12 &= -16 \Rightarrow 2t = -4 \Rightarrow \boxed{t = -2}, \boxed{s = 2}\end{aligned}$$

the point is $(0, 0, 7)$
(plug in $t = -2$ to $\vec{r}_2(t)$)

(d) Find an equation for the plane that contains both of the lines described in the problem.

Need $\vec{n}$ and a point in the plane

use $(0, 0, 7)$

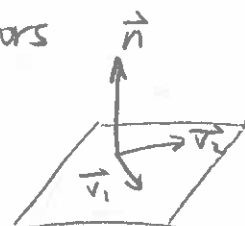
$$\text{so } \vec{r}_0 = \langle 0, 0, 7 \rangle$$

contains both direction vectors

$$\vec{v}_1 = \langle -2, 1, 3 \rangle$$

$$\vec{v}_2 = \langle -2, 1, 5 \rangle$$

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \langle 2, 4, 0 \rangle$$



$$(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$$

$$(\langle x, y, z \rangle - \langle 0, 0, 7 \rangle) \cdot \langle 2, 4, 0 \rangle = 0$$

$$\langle x, y, z - 7 \rangle \cdot \langle 2, 4, 0 \rangle = 0$$

$$\boxed{2x + 4y = 0}$$

2. Consider the two following planes: one with equation $4x - 5y + z = -2$ and the other which passes through the points $(1, 1, 1)$, $(0, 1, -2)$ and $(4, 2, -1)$.

(a) Find a normal vector to the first plane.

$$\langle 4, -5, 1 \rangle$$

(b) Find a scalar equation for the second plane.

$$\overrightarrow{QP} = \langle 1, 0, 3 \rangle, \quad \overrightarrow{PR} = \langle 3, 1, -2 \rangle$$

$$\vec{n} = \overrightarrow{QP} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 3 \\ 3 & 1 & -2 \end{vmatrix} = \hat{i}(-3) - \hat{j}(-2-9) + \hat{k} = \langle -3, 11, 1 \rangle$$

$$\vec{r}_0 = \langle 1, 1, 1 \rangle \Rightarrow (\langle x, y, z \rangle - \langle 1, 1, 1 \rangle) \cdot \langle -3, 11, 1 \rangle = 0$$

$$-3(x-1) + 11(y-1) + (z-1) = 0$$

(c) Find a point that lies on both planes.

$$\begin{cases} 4x - 5y + z = -2 \\ -3x + 11y + z = 9 \end{cases}$$

$$-3x + 11y + z = 9$$

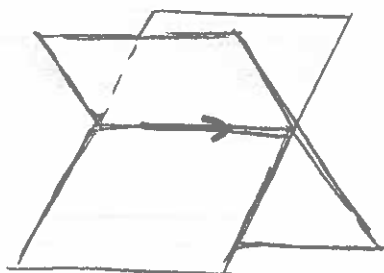
$$4x + z = -2$$

$$3x + z = -9$$

$$\Rightarrow 7x = -11 \Rightarrow x = -\frac{11}{7}, z = \frac{30}{7}$$

Assume $y=0 \Rightarrow$

(d) The two planes intersect in a line. Find the direction vector for the line of intersection of the two planes. 4.50



$\vec{v}$ is orthogonal to both normal vectors b/c $\vec{v}$ is in both planes

$$\Rightarrow \vec{v} = \vec{n}_1 \times \vec{n}_2$$

$$= \langle 4, -5, 1 \rangle \times \langle -3, 11, 1 \rangle$$

$$= \langle -16, -7, 29 \rangle$$

(e) Find parametric equations for the line of intersection of the two planes.

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

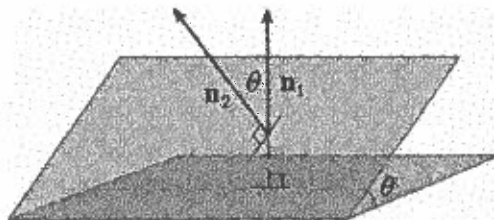
$$\text{point } \left(-\frac{11}{7}, 0, \frac{30}{7}\right) \text{ so } \vec{r}_0 = \left\langle -\frac{11}{7}, 0, \frac{30}{7} \right\rangle$$

$$\vec{r}(t) = \left\langle -\frac{11}{7}, 0, \frac{30}{7} \right\rangle + t \langle -16, -7, 29 \rangle$$

$$= \left\langle -16t - \frac{11}{7}, -7t, 29t - \frac{30}{7} \right\rangle$$

$$\begin{cases} x = -16t - \frac{11}{7} \\ y = -7t \\ z = 29t - \frac{30}{7} \end{cases}$$

Angles between planes: We define the angle θ between two planes to be the acute angle between their normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$.



$$\begin{aligned} x - 2y + 3z &= 2 \\ x - 2y + 3z &= 5 \end{aligned}$$

are parallel planes

We say that two planes are **parallel** if their normal vectors are parallel. We say that two planes are **perpendicular** if their normal vectors are orthogonal.

3. Do the planes $x + y + z = 1$ and $x - 2y + 3z = 1$ intersect? If yes, find the angle between the planes.

$$\begin{cases} x + y + z = 1 \\ x - 2y + 3z = 1 \end{cases}$$

$$\xrightarrow{z=0}$$

$$\begin{aligned} x + y &= 1 \\ x - 2y &= 1 \end{aligned}$$

$$3y = 0 \Rightarrow y = 0$$

$$\Rightarrow x = 1$$

intersect at $(1, 0, 0)$

$$\vec{n}_1 = \langle 1, 1, 1 \rangle, \quad \vec{n}_2 = \langle 1, -2, 3 \rangle$$

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \|\vec{n}_2\|}$$

$$= \frac{1 - 2 + 3}{\sqrt{3} \cdot \sqrt{1 + 4 + 9}} = \frac{2}{\sqrt{3} \sqrt{14}} = \frac{2}{\sqrt{42}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \approx 72^\circ$$

4. Find the point at which the line $\mathbf{r}(t) = \langle 2 - 2t, 3t, 1 + t \rangle$ intersects the plane $x + 2y - z = 7$. ~~What angle does the line make with the plane?~~ ← ignore

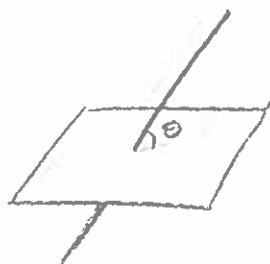
plug $x = 2 - 2t$, $y = 3t$, $z = 1 + t$ into the eq. of the plane

$$(2 - 2t) + 2(3t) - (1 + t) = 7$$

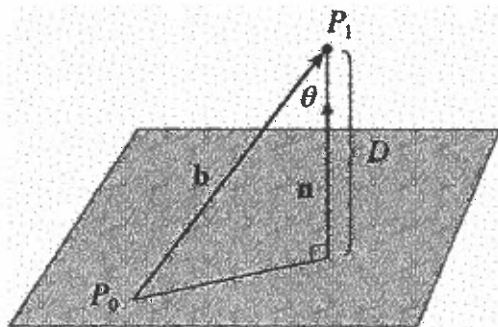
$$2 - 2t + 6t - 1 - t = 7$$

$$3t = 6$$

$$t = 2 \Rightarrow \text{point is } (-2, 6, 3)$$



Distance from a point to a plane: Consider a plane through the point P_0 with normal vector $\mathbf{n}$ and a point P_1 that is not on the plane.



The distance D from P_1 to the plane is equal to the length of the projection of $\mathbf{b}$ onto $\mathbf{n}$.

$$D = |\text{comp}_{\mathbf{n}} \mathbf{b}| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{\|\mathbf{n}\|}$$

5. Find the distance between the point $(1, 1, 5)$ and the plane $10x + 2y - 2z = 5$.

$$\vec{n} = \langle 10, 2, -2 \rangle$$

point: if $y=0, z=0, x=\frac{1}{2}$ $(\frac{1}{2}, 0, 0) = P_0$

$$\vec{b} = \overrightarrow{P_0 P_1} = \langle 1 - \frac{1}{2}, 1, 5 \rangle = \langle \frac{1}{2}, 1, 5 \rangle$$

$$D = \frac{|\vec{n} \cdot \vec{b}|}{\|\vec{n}\|} = \frac{|10 \cdot \frac{1}{2} + 2 + (-2) \cdot 5|}{\sqrt{10^2 + 4 + 4}} = \frac{|5 + 2 - 10|}{\sqrt{108}} = \frac{|-3|}{\sqrt{4 \cdot 27}}$$

$$= \frac{3}{2 \cdot 3\sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}}$$