

1. Find the length of the parametric curve $x = t^2$, $y = \frac{1}{3}t^3 - t$ for $0 \leq t \leq 3$.

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2. Find the equation of the tangent line to the parametric curve $x = t^3 + t$, $y = t^4 + t^2$ at $t = 1$.

$y = \frac{3}{2}x - 1$

3. Match each of the equations with a polar curve.

1. $r = \cos(\theta/3)$

I

4. $r = 3 \sin(4\theta)$

D

7. $r = \theta^2$

E

2. $r = 4 \sin(3\theta)$

B

5. $r = \sqrt{\theta}$

G

8. $r = \frac{1}{2} + \sin(\theta)$

H

3. $r = 1 + \cos(\theta)$

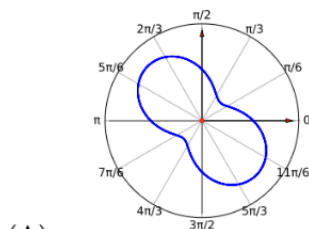
C

6. $r = \theta$

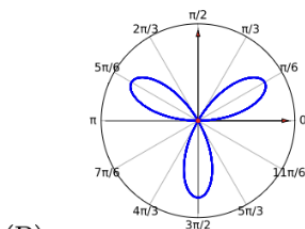
F

9. $r = 2 - \sin(2\theta)$

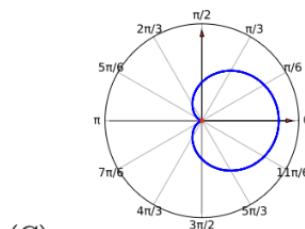
A



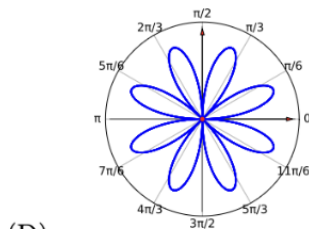
(A)



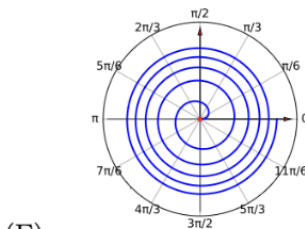
(B)



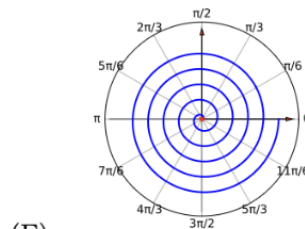
(C)



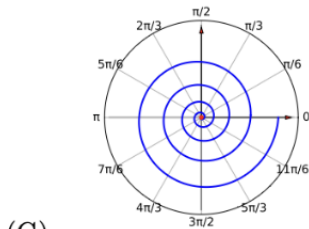
(D)



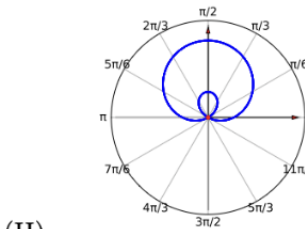
(E)



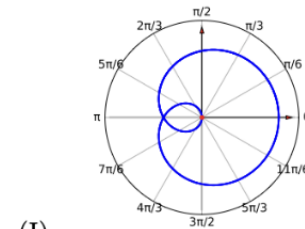
(F)



(G)



(H)



(I)

4. Find the area inside $r = 4 \cos \theta$ and outside $r = 2$.

$2\sqrt{3} + \frac{4\pi}{3}$

5. Determine if the sequence $a_n = \frac{n^2}{4n^2 - 3n + 4}$ converges or diverges.

converges to $1/4$

6. Suppose that $s_N = \frac{N^2 + 3N}{8N^2 + 1}$ is the N^{th} partial sum of the series $\sum a_n$.

(a) Determine if the series $\sum a_n$ converges or diverges.

(b) Find $\lim_{n \rightarrow \infty} a_n$.

7. Determine if the series converges or diverges. If it converges, find its sum.

(a) $\sum_{n=1}^{\infty} e^{-n}$

converges, value = $\frac{1}{e-1}$

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^5}}$

converges (can't compute value, oops!)

(c) $\sum_{n=3}^{\infty} (-1)^n \frac{2^{n-2}}{3^{n+1}}$

converges, value = $-\frac{2}{135}$

8. Use the Integral Test to determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$ converges or diverges.

converges (see solution for required details)

1. Find the length of the parametric curve $x = t^2$, $y = \frac{1}{3}t^3 - t$ for $0 \leq t \leq 3$.

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= \sqrt{(2t)^2 + (t^2 - 1)^2} dt$$

$$= \sqrt{4t^2 + (t^4 - 2t^2 + 1)} dt$$

$$= \sqrt{t^4 + 2t^2 + 1} dt$$

$$= \sqrt{(t^2 + 1)^2} dt$$

$$= t^2 + 1 dt$$

$$\begin{aligned} L = \int_0^3 ds &= \int_0^3 t^2 + 1 dt = \left[\frac{1}{3}t^3 + t \right]_0^3 \\ &= \frac{27}{3} + 3 = 12 \end{aligned}$$

2. Find the equation of the tangent line to the parametric curve $x = t^3 + t$, $y = t^4 + t^2$ at $t = 1$.

slope of tangent line :

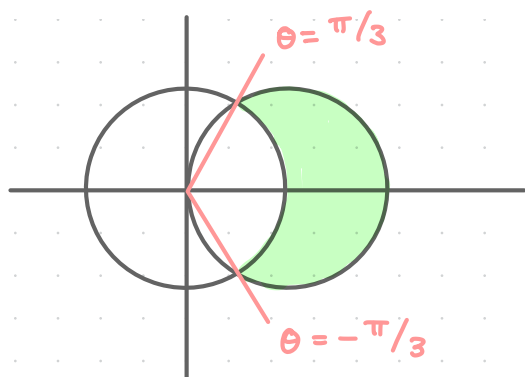
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t^3 + 2t}{3t^2 + 1}$$

$$\text{at } t=1, \quad \left. \frac{dy}{dx} \right|_{t=1} = \frac{4+2}{3+1} = \frac{6}{4} = \frac{3}{2}$$

point : at $t=1$, $x=2$ and $y=2$

point-slope form: $y-2 = \frac{3}{2}(x-2)$ or $y = \frac{3}{2}x - 1$

4. Find the area inside $r = 4 \cos \theta$ and outside $r = 2$.



intersection points:

$$4 \cos \theta = 2$$

$$\cos \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

$$A = \int_{-\pi/3}^{\pi/3} \frac{1}{2} ((4 \cos \theta)^2 - 2^2) d\theta$$

symmetry $\rightarrow$

$$= 2 \int_0^{\pi/3} \frac{1}{2} (16 \cos^2 \theta - 4) d\theta$$

$$= \int_0^{\pi/3} 16 \left(\frac{1 + \cos(2\theta)}{2} \right) - 4 d\theta$$

$$= \int_0^{\pi/3} 8 + 8 \cos(2\theta) - 4 d\theta$$

$$= \int_0^{\pi/3} 8 \cos(2\theta) + 4 d\theta$$

$$= \left[8 \cdot \frac{1}{2} \sin(2\theta) + 4\theta \right]_0^{\pi/3}$$

$$= 4 \sin\left(\frac{2\pi}{3}\right) + \frac{4\pi}{3} - 0$$

$$= 4\left(\frac{\sqrt{3}}{2}\right) + \frac{4\pi}{3}$$

$$= 2\sqrt{3} + \frac{4\pi}{3}$$

5. Determine if the sequence $a_n = \frac{n^2}{4n^2 - 3n + 4}$ converges or diverges.

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2}{4n^2 - 3n + 4} = \frac{1}{4} \quad \text{converges}$$

6. Suppose that $s_N = \frac{N^2 + 3N}{8N^2 + 1}$ is the N^{th} partial sum of the series $\sum a_n$.

(a) Determine if the series $\sum a_n$ converges or diverges.

(b) Find $\lim_{n \rightarrow \infty} a_n$.

$$(a) \sum a_n \overset{\text{by definition}}{=} \lim_{N \rightarrow \infty} s_N = \lim_{N \rightarrow \infty} \frac{N^2 + 3N}{8N^2 + 1} = \frac{1}{8}$$

the series converges to $\frac{1}{8}$

(b) Since the series converges, the terms must go to 0 (by Divergence Test).

$$\lim_{n \rightarrow \infty} a_n = 0.$$

7. Determine if the series converges or diverges. If it converges, find its sum.

$$(a) \sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n \quad r = \frac{1}{e} \text{ converges since } |r| < 1$$

$$(b) \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^5}} \quad \text{and value} = \frac{ye}{1 - 1/e} = \frac{1}{e-1}$$

$$(c) \sum_{n=3}^{\infty} (-1)^n \frac{2^{n-2}}{3^{n+1}}$$

$$(b) \sum_{n=1}^{\infty} \frac{1}{n^{5/3}} \quad p = \frac{5}{3} > 1 \text{ so converges}$$

$$(c) \sum_{n=3}^{\infty} (-1)^n \cdot \frac{2^n \cdot \frac{1}{4}}{3^n \cdot 3} = \sum_{n=3}^{\infty} \left(-\frac{2}{3}\right)^n \cdot \frac{1}{12} \quad r = -\frac{2}{3}, \text{ converges since } |r| < 1$$

$$\text{value} = \frac{\left(-\frac{2}{3}\right)^3 \cdot \frac{1}{12}}{1 - \left(-\frac{2}{3}\right)} = \frac{-\frac{8}{27} \cdot \frac{1}{12}}{\frac{5}{3}} = -\frac{2}{135}$$

8. Use the Integral Test to determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$ converges or diverges.

First, check the conditions!!! (f is continuous, positive and decreasing)

$$\text{Let } f(x) = \frac{1}{x(\ln x)^2}.$$

- f is continuous for $x > 1$ so continuous on $[2, \infty)$ ✓
- f is positive if $x > 0$, so positive on $[2, \infty)$ ✓
- f decreasing?

$$\begin{aligned} \frac{d}{dx} (x(\ln x)^2) &= 1 \cdot (\ln x)^2 + x \cdot 2 \ln x \cdot \frac{1}{x} \\ &= \underbrace{(\ln x)^2 + 2 \ln x}_{> 0 \text{ if } x > 1} \end{aligned}$$

Since f is 1 over an increasing function, f is decreasing. ✓

$$\begin{aligned} \int_2^{\infty} \frac{1}{x(\ln x)^2} dx &= \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x(\ln x)^2} dx \\ &= \lim_{t \rightarrow \infty} \int_{\ln 2}^{\ln t} \frac{1}{u^2} du \\ &= \lim_{t \rightarrow \infty} \int_{\ln 2}^{\ln t} u^{-2} du \\ &= \lim_{t \rightarrow \infty} \left[-u^{-1} \right]_{\ln 2}^{\ln t} \\ &= \lim_{t \rightarrow \infty} \left(-\frac{1}{\ln t} + \frac{1}{\ln 2} \right) = \frac{1}{\ln 2} \end{aligned}$$

$$\begin{aligned} u &= \ln x \\ du &= \frac{1}{x} dx \end{aligned}$$
$$\begin{aligned} x=2 &\Rightarrow u=\ln 2 \\ x=t &\Rightarrow u=\ln t \end{aligned}$$

Since the improper converges, the series also converges by the Integral Test.