

PFD, continued

ex: $\int \frac{2x^2 - x + 4}{x^3 + 4x} dx$

$\deg(\text{num}) < \deg(\text{denom})$
2 3
✓

PFD:

$\cancel{x(x^2+4)} \left(\frac{2x^2 - x + 4}{\cancel{x(x^2+4)}} \right) = \left(\frac{A}{x} + \frac{Bx+C}{x^2+4} \right) \cancel{x(x^2+4)}$

$$2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x$$

$x=0$:

$$4 = 4A + 0 \Rightarrow \boxed{A=1}$$

$x=1$:

$$5 = 5A + (B+C) \Rightarrow \boxed{B+C=0}$$

$\uparrow$
 $A=1$

$x=-1$:

$$7 = 5\underline{A} + (-B+C)(-1)$$

$$7 = 5 + B - C \Rightarrow \boxed{B - C = 2}$$

So $B + C = 0$

+ $B - C = 2$

$$\frac{2B = 2}{2B = 2} \Rightarrow \boxed{B=1} \text{ and } \boxed{C=-1}$$

$$\int \frac{2x^2 - x + 4}{x(x^2 + 4)} dx = \int \frac{1}{x} + \frac{x-1}{x^2+4} dx$$

$$= \ln|x| + \int \frac{x-1}{x^2+4} dx$$

$$= \ln|x| + \int \frac{x}{x^2+4} dx - \int \frac{1}{x^2+4} dx$$

$$\boxed{\begin{array}{l} u = x^2 + 4 \\ du = 2x dx \end{array}}$$

$$= \ln|x| + \int \frac{1}{u} \cdot \frac{1}{2} du - \int \frac{1}{x^2+4} dx$$

$$= \ln|x| + \frac{1}{2} \ln|u| - \int \frac{1}{x^2+4} dx$$

$$= \ln|x| + \frac{1}{2} \ln|x^2+4| - \frac{1}{2} \arctan\left(\frac{x}{2}\right) + C$$

Fact

$$\boxed{\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C}$$

Ex: $\int \frac{x^2+3x+9}{x^2(x^2+9)} dx$ $2 = \deg(\text{num}) < \deg(\text{denom}) = 4$ ✓

~~$\frac{x^2(x^2+9)}{x^2(x^2+9)}$~~ $\left(\frac{x^2+3x+9}{x^2(x^2+9)} \right) = \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C(x+D)}{x^2+9} \right) \cdot \cancel{x^2(x^2+9)}$

$x^2+3x+9 = \frac{A \cancel{x^2}(x^2+9)}{\cancel{x}} + \frac{B \cancel{x^2}(x^2+9)}{\cancel{x^2}} + (C(x+D))x^2$

$x^2+3x+9 = Ax(x^2+9) + B(x^2+9) + (Cx+D)x^2$

$x^2+3x+9 = Ax^3 + 9Ax + Bx^2 + 9B + Cx^3 + Dx^2$

Equate coeff:

$0 \cdot x^3 + 1 \cdot x^2 + 3x + 9 = (A+C)x^3 + (B+D)x^2 + 9Ax + 9B$

$A+C=0$

$B+D=1$

$9A=3$

$9B=9$

$\Rightarrow \boxed{A = 1/3}$
 $\Rightarrow \boxed{B = 1}$

$A+C=0 \Rightarrow C = -A = -1/3$ $\boxed{C = -1/3}$
 $B+D=1 \Rightarrow D = 1-B = 0$ $\boxed{D=0}$

$$\begin{aligned}
 \int \frac{x^2 + 3x + 9}{x^2(x^2 + 9)} dx &= \int \left(\frac{1/3}{x} + \frac{1}{x^2} + \frac{(-1/3)x + 0}{x^2 + 9} \right) dx \\
 &= \frac{1}{3} \int \frac{1}{x} dx + \int \frac{1}{x^2} dx - \frac{1}{3} \int \frac{x}{x^2 + 9} dx \\
 &\quad \quad \quad \begin{array}{l} u = x^2 + 9 \\ du = 2x dx \end{array} \\
 &= \frac{1}{3} \ln|x| + \int x^{-2} dx - \frac{1}{3} \int \frac{1}{u} \cdot \frac{1}{2} du \\
 &= \frac{1}{3} \ln|x| - x^{-1} - \frac{1}{3} \cdot \frac{1}{2} \ln|u| + C \\
 &= \frac{1}{3} \ln|x| - \frac{1}{x} - \frac{1}{6} \ln(x^2 + 9) + C
 \end{aligned}$$

If the degree of numerator is $\geq$ degree of denominator, use **polynomial long division**:

ex: $\int \frac{x^3 + x}{x - 1} dx$ deg(num) > deg(denom)

$$\begin{array}{r}
 \phantom{x-1 \overline{)} } x^2 + x + 2 \quad \leftarrow \text{result} \\
 x-1 \overline{) x^3 + 0 \cdot x^2 + x + 0} \\
 \underline{-) x^3 - x^2} \\
 \phantom{x-1 \overline{) }} x^2 + x + 0 \\
 \underline{-) x^2 - x} \\
 \phantom{x-1 \overline{) }} 2x + 0
 \end{array}$$

$$\begin{array}{r} -) \quad 2x - 2 \\ \hline 2 \end{array} \leftarrow \text{remainder}$$

$$\frac{x^3 + x}{x-1} = x^2 + x + 2 + \frac{2}{x-1}$$

$$\int x^2 + x + 2 + \frac{2}{x-1} dx$$

$$= \frac{1}{3} x^3 + \frac{1}{2} x^2 + 2x + 2 \ln |x-1| + C$$

ex: $\int \frac{x^2}{x^2-1} dx$

use poly. long division
if degrees are equal!

$$\begin{array}{r} x^2 - 1 \overline{) x^2 + 0x + 0} \\ -) \quad x^2 \quad \quad - 1 \\ \hline 1 \end{array} \begin{array}{l} \leftarrow \text{result} \\ \leftarrow \text{remainder} \end{array}$$

$$\frac{x^2}{x^2-1} = 1 + \frac{1}{x^2-1}$$

$$\int 1 + \frac{1}{x^2-1} dx = x + \int \frac{1}{x^2-1} dx$$

$$= x + \int \underbrace{\frac{1}{(x+1)(x-1)}}_{\text{PFD}} dx$$

$$\cancel{(x+1)(x-1)} \left(\frac{1}{\cancel{(x+1)(x-1)}} \right) = \left(\frac{A}{x+1} + \frac{B}{x-1} \right) (x+1)(x-1)$$

$$1 = A(x-1) + B(x+1)$$

$$\underline{x=1}: \quad 1 = 0 + 2B \quad \Rightarrow \quad B = \frac{1}{2}$$

$$\underline{x=-1}: \quad 1 = -2A + 0 \quad \Rightarrow \quad A = -\frac{1}{2}$$

$$\begin{aligned} \int \frac{(-1/2)}{x+1} + \frac{(1/2)}{x-1} dx &= -\frac{1}{2} \int \frac{1}{x+1} dx + \frac{1}{2} \int \frac{1}{x-1} dx \\ &= -\frac{1}{2} \ln|x+1| + \frac{1}{2} \ln|x-1| + C \end{aligned}$$

Answer: $x - \frac{1}{2} \ln|x+1| + \frac{1}{2} \ln|x-1| + C$

↗
don't forget
first term
from before!