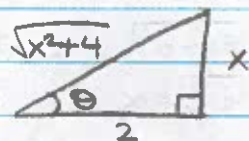


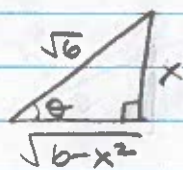
#1) a. $\int \frac{x^2}{\sqrt{x^2+4}} dx$

since $\sqrt{x^2+4}$ appears,
use $x = 2 \tan \theta$



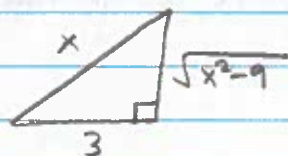
b. $\int \frac{1}{\sqrt{6-x^2}} dx$

since $\sqrt{6-x^2}$ appears,
use $x = \sqrt{6} \sin \theta$



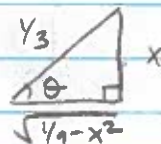
c. $\int \frac{x^2}{(x^2-9)^{3/2}} dx$

since $(x^2-9)^{3/2} = (\sqrt{x^2-9})^3$
appears, use $x = 3 \sec \theta$



$$\begin{aligned} \text{d. } \int x^3 \sqrt{1-9x^2} dx &= \int x^3 \sqrt{9(1/9 - x^2)} dx \\ &= \int x^3 \sqrt{9} \cdot \sqrt{1/9 - x^2} dx \\ &= \int 3x^3 \sqrt{1/9 - x^2} dx \end{aligned}$$

Since $\sqrt{1/9 - x^2}$ appears, use $x = \frac{1}{3} \sin \theta$



(Alternatively, $\int x^3 \sqrt{1-9x^2} dx = \int x^3 \sqrt{1-(3x)^2} dx$
so use $3x = \sin \theta$ w/ triangle )

#2) a. $f(x) = \frac{x+1}{x^3-x} = \frac{x+1}{x(x^2-1)} = \frac{x+1}{x(x+1)(x-1)}$

form of PFD: $\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$

b. $g(x) = \frac{x^3-1}{(x+2)^2(x^2+9)}$

irreducible quadratic

form of PFD: $\frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{(x+D)}{x^2+9}$

#3) a. $\int_0^e \frac{\ln x}{x} dx$ is improper because $\frac{\ln x}{x}$ is not continuous at $x=0$

Write as a limit: $\int_0^e \frac{\ln x}{x} dx = \lim_{t \rightarrow 0^+} \int_t^e \frac{\ln x}{x} dx$

$u = \ln x$	$x = t \Rightarrow u = \ln t$
$du = \frac{1}{x} dx$	$x = e \Rightarrow u = \ln e = 1$

$$= \lim_{t \rightarrow 0^+} \int_{\ln t}^1 u du$$

$$= \lim_{t \rightarrow 0^+} \left(\frac{1}{2} u^2 \right) \Big|_{\ln t}^1$$

$$= \lim_{t \rightarrow 0^+} \left(\frac{1}{2} - \frac{1}{2} (\ln t)^2 \right)$$

$$= \frac{1}{2} - \frac{1}{2} \lim_{t \rightarrow 0^+} (\ln t)^2$$

$$= \frac{1}{2} - \frac{1}{2} (\infty) = -\infty$$

b. $\int_1^5 \frac{x}{3-x} dx$ has a discontinuity at $x=3$

$$\int_1^5 \frac{x}{3-x} dx = \underbrace{\int_1^3 \frac{x}{3-x} dx}_{(1)} + \underbrace{\int_3^5 \frac{x}{3-x} dx}_{(2)}$$

Compute ① and ② separately.

First compute the indefinite integral

$\int \frac{x}{3-x} dx$ using polynomial division

$$\begin{array}{r} -1 \\ -x+3 \overline{) x+0} \\ \underline{-x+3} \\ 3 \end{array}$$

$$\text{so } \frac{x}{3-x} = -1 + \frac{3}{3-x}$$

$$\begin{aligned} \int \frac{x}{3-x} dx &= \int -1 + \frac{3}{3-x} dx \\ &= -x + 3 \int \frac{1}{3-x} dx \\ &= -x - 3 \int \frac{1}{x-3} dx \\ &= -x - 3 \ln|x-3| + C \end{aligned}$$

$$\begin{aligned} \text{For ①: } \int_1^3 \frac{x}{3-x} dx &= \lim_{t \rightarrow 3^-} \int_1^t \frac{x}{3-x} dx \\ &= \lim_{t \rightarrow 3^-} \left(-x - 3 \ln|x-3| \right) \Big|_1^t \\ &= \lim_{t \rightarrow 3^-} \left(-t - 3 \ln|t-3| - (-1 - 3 \ln(3)) \right) \\ &= 3 - 3 \lim_{t \rightarrow 3^-} (\ln|t-3|) + 1 + 3 \ln(3) \\ &= 4 + 3 \ln(3) - 3 \cdot (-\infty) \\ &= \infty \quad \text{so 'diverges'} \end{aligned}$$

Since ① diverges, the entire integral diverges

c. $\int_{-2}^{\infty} \frac{1}{x^3} + 1 \, dx$

is improper because there's a discontinuity at $x=0$ and the integral is over an infinite interval.

$$\int_{-2}^{\infty} \frac{1}{x^3} + 1 \, dx = \int_{-2}^0 \frac{1}{x^3} + 1 \, dx + \int_0^1 \frac{1}{x^3} + 1 \, dx + \int_1^{\infty} \frac{1}{x^3} + 1 \, dx$$

#4) a. $\int \frac{x}{x^2-4} \, dx$ u-sub w/ $u = x^2-4$

b. $\int \frac{1}{x^2-4} \, dx = \int \frac{1}{(x+2)(x-2)} \, dx$ PFD

c. $\int \frac{1}{x^2+4} \, dx = \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + C$

d. $\int \frac{x^3}{x^2-4} \, dx$ polynomial division and PFD

e. $\int_1^2 \frac{1}{(x^2-1)^{3/2}} \, dx$ trig sub

f. $\int \frac{x^3+2x+x}{x^2+1} \, dx$ polynomial division

g. $\int_0^{\pi/2} \cos^5 x \, dx = \int_0^{\pi/2} (1-\sin^2 x)^2 \cos x \, dx$

h. $\int x \arctan x \, dx$ IBP w/ $u = \arctan x$ $dv = x \, dx$

i. $\int \frac{e^{1/x}}{x^2} \, dx$ u-sub with $u = 1/x$

$$j. \int \sec^3 x \tan x \, dx = \int \sec^2 x \cdot \sec x \tan x \, dx$$

u-sub w/ $u = \sec x$

$$k. \int e^{2x} \sin x \, dx$$

IBP (boomerang)

$$l. \int_1^e \frac{\ln x}{x^2} \, dx$$

IBP w/ $u = \ln x$, $dv = \frac{1}{x^2} dx$

$$\#5) a. \int \frac{x^3}{\sqrt{x^2-4}} \, dx = \int \frac{x^2 \cdot x}{\sqrt{x^2-4}} \, dx$$

$$u = x^2 - 4 \Rightarrow x^2 = u + 4$$

$$du = 2x \, dx$$

$$\frac{1}{2} du = x \, dx$$

$$= \int \frac{(u+4)}{\sqrt{u}} \, du$$

$$= \int \frac{u}{\sqrt{u}} + \frac{4}{\sqrt{u}} \, du$$

$$= \int u^{1/2} + 4u^{-1/2} \, du$$

$$= \frac{2}{3} u^{3/2} + 4(2u^{1/2}) + C$$

$$= \frac{2}{3} (x^2-4)^{3/2} + 8\sqrt{x^2-4} + C$$

$$\#5) b. \int e^{2x} \ln(1+e^x) dx = \int e^x e^x \ln(1+e^x) dx$$

$$u = 1+e^x \Rightarrow e^x = u-1$$

$$du = e^x dx$$

$$= \int (u-1) \ln(u) du$$

$$w = \ln u \quad dv = u-1 du$$

$$dw = \frac{1}{u} du \quad v = \frac{1}{2}u^2 - u$$

$$= \ln u \left(\frac{1}{2}u^2 - u \right) - \int \left(\frac{1}{2}u^2 - u \right) \frac{1}{u} du$$

$$= \left(\frac{1}{2}u^2 - u \right) \ln u - \int \frac{1}{2}u - 1 du$$

$$= \left(\frac{1}{2}u^2 - u \right) \ln u - \frac{1}{2} - \frac{1}{2}u^2 + u + C$$

$$= \left(\frac{1}{2}u^2 - u \right) \ln u - \frac{1}{4}u^2 + u + C$$

$$= \left(\frac{1}{2}(1+e^x)^2 - (1+e^x) \right) \ln(1+e^x)$$

$$- \frac{1}{4}(1+e^x)^2 + (1+e^x) + C$$

$$\#6) a. \int_2^{\infty} \frac{x}{\sqrt{x^4-x}} dx$$

$$\text{compare to } \frac{x}{\sqrt{x^4}} = \frac{x}{x^2} = \frac{1}{x}$$

$$\int_1^{\infty} \frac{1}{x} dx \text{ diverges since } p=1 \leq 1$$

$$\left(\Rightarrow \int_2^{\infty} \frac{1}{x} dx \text{ also diverges} \right)$$

$$\text{Want } \frac{1}{x} \leq \frac{x}{\sqrt{x^4-x}}$$

$$0 \leq \sqrt{x^4-x} \leq x^2$$

$$x^4-x \leq x^4 \text{ which is true.}$$

$$\text{By the Comparison, } \int_2^{\infty} \frac{x}{\sqrt{x^4-x}} dx \text{ also diverges.}$$

#6) b. $\int_1^{\infty} \frac{\sin^2(x)}{\sqrt{x^3}} dx$

compare to $\frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}}$
 $\int_1^{\infty} \frac{1}{x^{3/2}} dx$ converges since
 $p = 3/2 > 1$.

Want: $\frac{\sin^2 x}{x^{3/2}} \leq \frac{1}{x^{3/2}}$

$\sin^2 x \leq 1$ which is true because
 $\sin x \leq 1$

By the Comparison Theorem, $\int_1^{\infty} \frac{\sin^2 x}{\sqrt{x^3}} dx$
also converges.