GIT & VGIT

§ GIT

- GIT was motivated by the following question:
  If we are given a variety X with an algebraic group G acting on it,
  is the orbit space X/G also a variety?
- The answer is no in general.
  E.g. Consider $G = \mathbb{C}^* \curvearrowright X = \mathbb{C}$
  Then $X/G = \{0, \mathbb{C}^*\}$
  However, $\mathbb{C}^*$ is not closed and its closure $\overline{\mathbb{C}^*} = \mathbb{C}$ contains 0.
  We cannot separate these two orbits algebraically.

- To address this, we first restrict G to a special class of groups

Def: $G \subseteq GL_n(k)$ is reductive if it is a linear algebraic group with trivial unipotent radical.

Ex: SL_n, GL_n, O_n, SO_n, $Sp_{2n}(k)$.
Non-Ex: $\mathbb{G}_a := (\mathbb{C}, +)$

- Then we need to make sense of "nice" quotients.

First, we want a universal quotient in the category of varieties

Def: A categorical quotient is a variety Y with a G-invariant morphism $f: X \to Y$ which satisfies the universal property.
That is,
1) $\forall x \in X, g \in G,\ f(g \cdot x) = f(x)$
2) $\forall (Y', f': X \to Y')$
$\exists!\ Y \to Y'$ s.t.

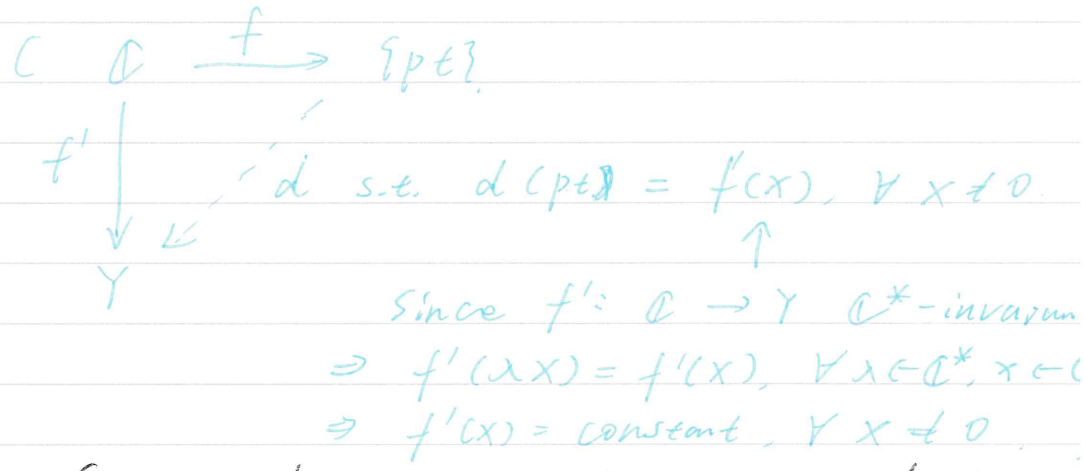
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow^{f'} & \downarrow \exists! \\ & & Y' \end{array}$$

This is the best possible quotient in the category of varieties, but it does not necessarily have good geometric / topological properties

E.g. Consider $G = \mathbb{C}^* \curvearrowright X = \mathbb{C} = \operatorname{Spec} \mathbb{C}[x]$

- The categorical quotient is
$$Y = \operatorname{Spec}(\mathbb{C}[x]^{\mathbb{C}^*}) = \operatorname{Spec}(\mathbb{C}) = \{pt\}$$
$h \in \mathbb{C}[x]^{\mathbb{C}^*}$ iff $h(\lambda x) = h(x)$, $\forall \lambda \in \mathbb{C}^*$ iff h = constant

- One can check that $f: \mathbb{C} \to \{pt\}$ is universal and $\mathbb{C}^*$-~~invariant~~ invariant.



- So we have one pt corresponding to two orbits $\mathbb{C}^*$ and $\{0\}$.
Not ideal.

Def: $f: X \to Y$ is a good quotient for the action of G on X if

1) f is G-invariant.

2) $\mathcal{O}_Y = (f_* \mathcal{O}_X)^G$ $\Rightarrow$ f dominant
(i.e. The regular fcns on Y are precisely the G-invariant regular fcns on X)
($\forall U \subseteq Y$ $\mathcal{O}_Y(U) = \mathcal{O}_X(\pi^{-1}(U))$

So Y has quotient topology

3) f is a closed map on G-invariant closed subset $W \subseteq X$
(i.e. $W \subseteq X$ G-invariant and closed $\Rightarrow f(W)$ closed.)

2) + 3) $\Rightarrow$ f is surjective.

3) $\Rightarrow$ $\forall U \subseteq Y$ open aff $f^{-1}(U) \subseteq X$ open

4) If W_1 and W_2 are G-invariant + closed + disjoin
then $f(W_1)$ and $f(W_2)$ are disjoint.

3) + 4) $\Rightarrow$ $f(W_1)$ and $f(W_2)$ disjoint.

5) f is affine.
(i.e. The preimage of every affine open subset is affine.)

Good quotient inherits the algebraic structure from X (so Y is a variety) but its fibers may contain more than one orbit.

Def: $f: X \to Y$ is a geometric quotient if 1) f is a good quotient
2) $\forall x \in X$, $\pi^{-1}(\pi(x)) = G \cdot x$

$\downarrow$

Every orbit corresponds to exactly one pt in the quotient.
Geometric quotient gives us a genuine moduli space.

Geometric. q. $\Rightarrow$ Good. q. $\Rightarrow$ Categorical. q.

For the affine case:

Given reductive $G \curvearrowright$ affine variety X

Then

1) If $X = \operatorname{spec} R$
then the categorical quotient exises
and $Y := \operatorname{Spec} R^G$

2) The pts in Y correspond to equivalence classes of $G \cdot x$ $x \in X$
where $G \cdot x_1 \sim G \cdot x_2$ iff $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset$

3) ~~Exis~~ For each equivalence class of orbits
$\exists$! closed orbit, minimal orbit,
which lies in the closure of every other
orbit in the class.

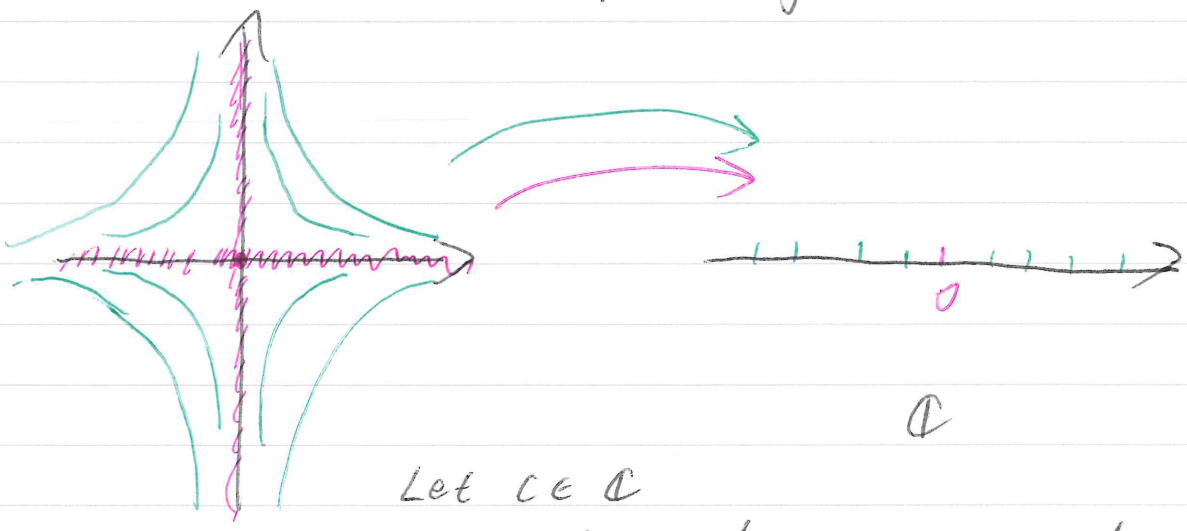
Ex:

① $\mathbb{C}^* \curvearrowright \mathbb{C}^2$ s.t. $(x, y) \mapsto (tx, t^{-1}y)$

Then $R = \mathbb{C}[x, y]$

$R^G = \mathbb{C}[xy]$

So $\mathbb{C}^2 /\!/ \mathbb{C}^* = \operatorname{spec} \mathbb{C}[xy] = \mathbb{C}$



$\mathbb{C}$

Let $c \in \mathbb{C}$

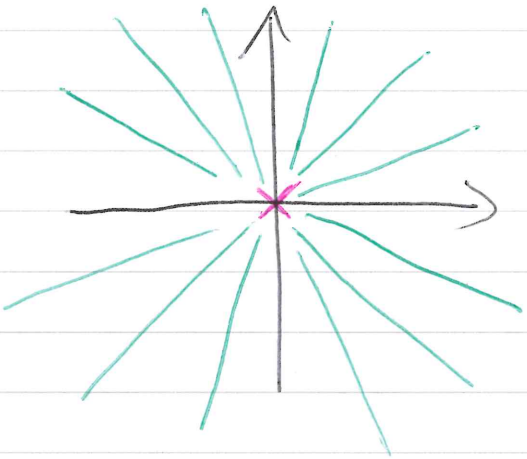
$\mathbb{C}^2$ · For $c \neq 0$, the preimage of c is
the closed orbit $xy = c$
· For $c = 0$, the preimage of 0 is

- $\{(0,0)\}$ is the minimal orbit.

② $\mathbb{C}^* \curvearrowright \mathbb{C}^2$ s.t. $(x,y) \mapsto (tx, ty)$

- Then $R^{\mathbb{C}^*} = \mathbb{C}$
  $\Rightarrow \mathbb{C}^2 /\!/ \mathbb{C}^* = \operatorname{Spec} \mathbb{C} = \{pt\}$

- For $(x_0, y_0) \neq (0,0)$,
  $\mathbb{C}^* \cdot (x_0, y_0) = \{(x,y),\ y = cx,\ (x,y) \neq (0,0)\}$.
  For $(0,0)$, $\mathbb{C}^* \cdot (0,0) = \{(0,0)\}$



The only closed orbit is $\mathbb{C}^* \cdot (0,0)$.

- Now, removing $\{x = 0\}$.
  Then $R = \mathbb{C}[x, \frac{1}{x}, y]$
  $R^{\mathbb{C}^*} = \mathbb{C}[\frac{y}{x}]$
  $\Rightarrow (\mathbb{C}^2 \setminus \mathbb{C}) /\!/ \mathbb{C}^* = \operatorname{spec} \mathbb{C}[\frac{y}{x}] = \mathbb{C}$

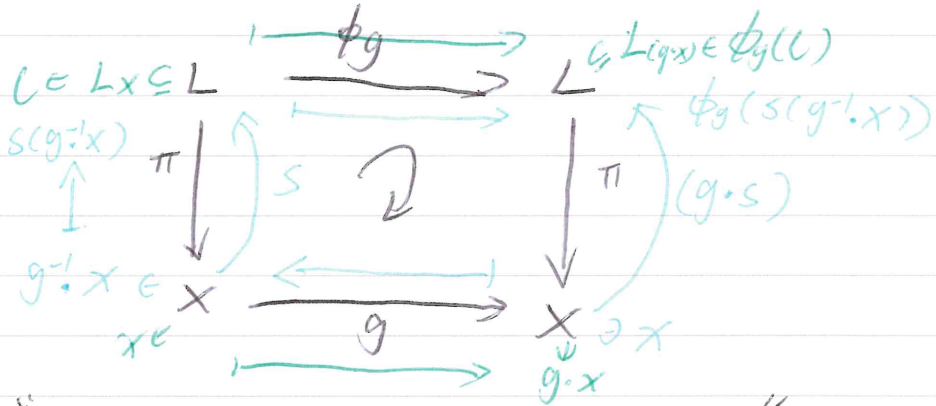
Now all the orbits are of the form
$\mathbb{C}^* \cdot (x_0, y_0) = \{(x,y) \mid y = cx\}$.
All the orbits are closed.

Enlightening: Removing "bad orbit" can result in a bigger quotient.

For the projective case:

- Given a ~~star~~ polarized variety (X, L) and L is an ample linearisation. G reduces

i.e. $\forall g \in G$, $\exists\ \phi_g : L \to L$ s.t.



i.e. "$\ell \in L_x \iff \phi_g(\ell) \in L_{g\cdot x}$"

$\iff$ $L_x \to L_{g\cdot x}$ is a linear isomorphism

- Then we have a G-action on $H^0(X, rL)$ s.t.

$$G \times H^0(X, rL) \longrightarrow H^0(X, rL)$$
$$(g, s) \longmapsto g\cdot s$$

where $(g\cdot s)(x) = \phi_g(s(g^{-1}x))$

This induces G-action on $\bigoplus_{r\geq 0} H^0(X, rL)$ $\;\;!!\;\;$ $R(X, L)$

In particular, G-action preserves the graded algebra structure of $R(X, L)$ and $R(X, L)^G$ is finitely generated.

The multiplication of $R(X, L)$ is G-equivariant.

- let $s \in H^0(X, dL)$, $t \in H^0(X, eL)$
  Then $s \otimes t \in H^0(X, (d+e)L)$
  and $g\cdot(s\otimes t) = (g\cdot s)\otimes(g\cdot t)$
  (by def of linearisation)

- So the ring structure of $R(X, L)$ is preserved by the G-action

- In particular, $g \cdot R(X,L)_d \subseteq R(X,L)_d$
  So the grading is also preserved
  i.e. $R(X,L)$ is a G-equivariant graded algebra. ⊣

T. Since L is ample, $R(X,L)$ is finitely generated.

- Recall the Nagata's Thm:
  If $\begin{cases} \cdot\ G$ a reductive algebraic gp over k with $\mathrm{char}(k)=0$
  $\cdot$ A finitely generated k-algebra on which G acts by algebra automorphism

  then A^G is a finitely generated k-algebra. ⊣

- This induces a G-action on $\mathrm{Spec}\, R(X$
  The affine cone $\hat{X}$ o

T. If $G \curvearrowright A$ by any automorphism X
  (i.e. $\forall g \in G$, $\phi_g : A \to A$ s.t.
  ~~1) $\phi_g(1) = 1$~~ $a \mapsto g \cdot a$
  ~~2) $\phi_g(ab) = \phi_g(a)\phi_g(b)$, $a,b \in A$~~

  1) $\phi_{gh} = \phi_g \circ \phi_h$, $g,h \in G$
  2) $\phi_e = \mathrm{Id}_A$, $e \in G$ identity

  then $G \curvearrowright \mathrm{Spec}\, A$ by scheme automorphis
  (i.e. $G \times \mathrm{Spec}\, A \longrightarrow \mathrm{Spec}\, A$
  $(g,p) \longmapsto g \cdot p := \phi_g^{-1}(p)$)

  ⇒ The morphism of locally ringed spac
  follows.

- Now, we reduce to the affine case. The categorical quotient of $\tilde{X}/G$ is
$$\tilde{Y} := \operatorname{Spec} R(X,L)^G$$
- So the quotient of X should be
$$\operatorname{Proj} R(X,L)^G =: Y$$

- Note that the vertex $0 \in \tilde{Y}$ does not give a pt in Y, so we must remove the G-orbits in $\tilde{X}$ equivalent to 0 i.e. $\overline{G \cdot \tilde{x}} \ni 0$

$\Downarrow$

The nullcone of X, $N(X)$

$$N(X) := \{ x \mid 0 \in \overline{G \cdot \tilde{x}} \}$$
$$= \left\{ x \mid s(x) = 0, \ \forall s \in H^0(X, mL)^G, \ m > 0 \right\}$$

X^{us}, X^{ss} & X^s

Let $x \in X$

- $x \in X^{us}$ if $0 \in \overline{G \cdot \tilde{x}}$, $\tilde{x}$ lift of x in affine cone $\tilde{X}$ of X
- $x \in X^{ss}$ if $x \in X \setminus X^{us}$ i.e. $0 \notin \overline{G \cdot \tilde{x}}$
- $x \in X^s$ if $x \in X^{ss}$, $G \cdot x$ closed & G_x finite.

Thm: Given G reductive $\curvearrowright (X, L)$ polarised and L ample linearisation.
Then

1) X^{ss} is open in X
and its categorical quotient is
$$X /\!\!/_L G := \operatorname{Proj} R(X, L)^G$$
↑ Good quotient

2) Points of $X /\!\!/_L G$ are G-orbits modulo the $\sim$ s.t.
$G \cdot x_1 \sim G \cdot x_2$ iff $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset$
(i.e. Two semistable pts mapped $\leftrightarrow$ the same pt in $X /\!\!/_L G$ if all the homogeneous invariant fcns take the same values on them.)

All non-closed orbits degenerate $\Leftarrow$ to a polystable orbit

3) In every equivalence class $\exists$! closed ~~orbits~~ orbit contained in the closure of every other orbit
↑ polystable orbit.

4) X^s is open in X^{ss} and its geometric quotient exists.
(Pts of X^s/G are G-orbits of X^s)

Ex: Given $(\mathbb{P}^1, \mathcal{O}(1))$

① Consider $\mathbb{C}^* \curvearrowright \mathbb{P}^1$ s.t. $[x:y] \mapsto [tx : t^{-1}y]$
$X^{us} = \{xy = 0\} = \{[1:0], [0:1]\}$
$X^{ss} = X^s = \mathbb{P}^1 \setminus \{[1:0], [0:1]\}$
$X /\!\!/ G = \{pt\}$.

② $\mathbb{C}^* \curvearrowright \mathbb{P}^1$ s.t. $[x:y] \mapsto [tx : ty]$
$X^{ss} = \emptyset$
$X /\!\!/ G = \emptyset$

§ GIT quotient by a torus action

① First we consider the case that the red. gp G is a torus $T = (\mathbb{G}_m)^r$
Then $T \curvearrowright R = R(X, L)$ is diagonalizable and R decomposes into

$$R = \bigoplus_{m \in M} R_m$$

where $M := \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}^r$ the character lattice.

and $R_m := \{ f \in R \mid \forall t \in T,\ t \cdot f = \chi_m(t) f \}$ the weighted subspace of wt m

$\chi_m \in M$ s.t. $\chi_m(t) = t_1^{m_1} \cdot \ldots \cdot t_r^{m_r}$

Recall that $\forall f \in R$, f decomposes ! into a finite sum $f = \sum_{m \in M} f_m$, $f_m \in R_m$
and Multiplication in R respects the weight i.e. $f \in R_m$ and $g \in R_{m'} \Rightarrow fg \in R_{m+m'}$

So $R^T = \{ f \in R \mid \forall t \in T,\ t \cdot f = f \} = R_0$
$\Rightarrow X /\!/ T = \mathrm{Proj}\, R_0$

② · Now let us consider the polarized toric variety (X, L) of dim n w.r.t. the big torus $H = (\mathbb{G}_m)^n$
L is an ample H-linearisation.

· Then (X, L) corresponds to a polytope Q, where

$$Q := \mathrm{Conv}\{ m \in M \mid H^0(X, L)_m \neq 0 \}$$

- To build the graded way.
embed $Q \subset M_{\mathbb{R}}$ into $\mathbb{R} \oplus M_{\mathbb{R}}$.

$$\begin{array}{ccc} Q & \hookrightarrow & \mathbb{R} \oplus M_{\mathbb{R}} \\ m & \mapsto & (1, m) \end{array}$$

Then $(1, Q) = \{(1, m) \in \mathbb{Z} \oplus M_{\mathbb{R}} \mid m \in Q\}$.
and the cone over $(1, Q)$ is
$\text{Cone}(1, Q) = \{\lambda(1, m) \mid \lambda \geq 0,\ m \in Q\}$.

pts with integer coordinates

- The lattice pts of $\text{Cone}(1, Q)$ are
$\mathbb{Z}^{n+1} \cap \text{Cone}(1, Q)$
$:= \{(k, m) \in \mathbb{Z}_{\geq 0} \times M \mid m \in kQ\}$.

- Clearly $(0, 0) \in \mathbb{Z}^{n+1} \cap \text{Cone}(1, Q)$.
and $\mathbb{Z}^{n+1} \cap \text{Cone}(1, Q)$ is closed under
+
$\Rightarrow \mathbb{Z}^{n+1} \cap \text{Cone}(1, Q)$ is a semigp.

- Let $S := \mathbb{Z}^{n+1} \cap \text{Cone}(1, Q)$.
Define the semigp algebra of S

$$k[S] := \bigoplus_{(k,m) \in S} k \cdot x_{(k,m)}$$

same k

as vector space over k with
multiplication $x_{(k,m)} \cdot x_{(k',m')} = x_{(k+k', m+m')}$

- $k[S]$ has a natural $\mathbb{Z}_{\geq 0}$-grading by the first coordinate k.

$$k[S] = \bigoplus_{k \geq 0} R_k, \quad R_k = \bigoplus_{m \in kQ \cap M} k \cdot x_{(k,m)}$$

same k

- Hence, one can identify
$$R(X, L) = k[S]$$

- So $X = \mathrm{Proj}\, R(X, L) = \mathrm{Proj}\, k[S]$
  $L = \mathcal{O}_{\mathrm{proj}\, k[S]}(1)$

③ Let $T \subset H$ subtorus of rank r.

- Then $\exists$
$$\varphi : M_H \twoheadrightarrow M_T \quad \text{(projection)}$$

- Also R can be decomposed into
$$R = \bigoplus_{m \in M_T} R^{(T)}_m, \quad R^{(T)}_m := \bigoplus_{x \in \varphi^{-1}(m)} R_x$$

- $R^T = R^{(T)}_0$
$$= k[\, \{(k, m) \in S \mid \varphi(m) = 0\}\,]$$

$$\exists\!\!\exists$$
$$\mathbb{Z}^{1+N'} \cap \mathrm{Cone}(1, Q_0)$$
where $N' := \dim(\varphi^{-1}(0))$
$Q_0 := Q \cap \varphi^{-1}(0)$

- So $X /\!\!/_L T := \mathrm{Proj}\,(R^T)$
$$= \mathrm{Proj}\,(k[\mathbb{Z}^{1+N'} \cap \mathrm{Cone}(1, Q_0)])$$

§ VAIT

VAIT : Tracking how the GIT quotient changes when the linearisation changes.

We can either fix L and varying a for aL or vary L within $\mathrm{Pic}X \otimes \mathbb{Q}$

We only ~~toke~~ discuss the first case, since the second is an analogue.

We replace L by aL, $a>0$

① · The homogeneous coordinate ring (w.r.t. L) is $R = \bigoplus_{d\geq 0} H^0(X, dL)$

· The .. (w.r.t. aL) is $R^{(a)} = \bigoplus_{d\geq 0} H^0(X, adL)$

↑

Veronese subring

· One can prove $\mathrm{Proj}\, R = \mathrm{Proj}\, R^{(a)}$

So the variety X is unchanged

$\mathrm{Proj}\, R = \{ P \in R \mid P \text{ homogeneous prime ideal s.t. } P \not\supseteq R_+ \}$

$\mathrm{Proj}\, R^{(a)} = \{ \ldots \}$

$\forall\, P \in \mathrm{Proj}\, R$, $P^{(a)} := P \cap R^{(a)}$ does not contain R_+

$\forall\, q \in \mathrm{Proj}\, R^{(a)}$, qR does not contain R_+

So $X/\!/T$ is unchanged

- This also motivates fractional linearisation

- Let $s \in H^0(X, L)$
  Then $s^{\otimes a} \in H^0(X, L^{\otimes a})$

- Recall $G \curvearrowright H^0(X, L)$ by
  $g(s(x)) \in L_{(g \cdot x)}$,

- Define $g(s^{\otimes a}(x)) = (g(s(x)))^{\otimes a}$
  So we get the $G \curvearrowright H^0(X, L^{\otimes a})$

- In HM-weight,
  ~~Let x_0~~ given $\rho : \mathbb{C}^* \to G$ 1-PS.
  Let $x_0 := \lim_{t \to 0} \rho(t) \cdot x$

  If $z \in L_{x_0}$ and $\rho(t) \cdot z = t^\lambda z$
  then $z^{\otimes a} \in (L_{x_0})^{\otimes a}$ ~~and~~
  and $\rho(t) \cdot z^{\otimes a} = (t^\lambda z)^{\otimes a} = t^{\lambda a} z^{\otimes a}$

- So the HM-weight $\mu(X, \rho)$ w.r.t. L is $-\lambda$, whereas $\mu(X, \rho)$ w.r.t. $L^{\otimes}$ is $-\lambda a$.

(2) For the polytopes:

$$Q \rightsquigarrow aQ$$

$Q_0 = Q \cap \ell^{-1}(0) \rightsquigarrow$ The parallel slice $Q_c = Q \cap \ell^{-1}(c)$
where $\ell : Q \to M_T \otimes \mathbb{C}$
$(d, m) \mapsto m$

③ We can also ~~fixe~~ fix a line bundle L and change the linearisation:

- Consider $\mathbb{C}^* \curvearrowright (X, L)$ polarised. Then $\mathbb{C}^* \curvearrowright H^0(X, L)$ and we can choose the coordinates s.t. $\mathbb{C}^* \curvearrowright H^0(X, L)$ diagonally

$$\begin{pmatrix} t^{\lambda_0} & & \\ & \ddots & \\ & & t^{\lambda_N} \end{pmatrix}$$

- Recall that if (X, L) is polarised toric variety, then every line bundle on X is ~~due~~ linearizable and two linearization differ by $m \in M$ where $M := \mathrm{Hom}(T, \mathbb{G}_m)$ character lattice

- So we can change the linearisation s.t. $T \curvearrowright H^0(X, L)$ by

$$\begin{pmatrix} t^{\lambda_0 + c} & & \\ & \ddots & \\ & & t^{\lambda_N + c} \end{pmatrix}$$

- Let $\mathcal{L}$ denote the original linearisation. Then the new linearisation is $\mathcal{L}' := \mathcal{L} \otimes \chi_m$, for some $m \in M$.

~~a~~ $\forall\, t \in T$, $v \in L_x$,

$$t \cdot_{\mathcal{L}'} v = \chi_m(t) \cdot (t \cdot_{\mathcal{L}} v)$$

· Note that this does not change the $T \curvearrowright X$, since X is projectiv

Let $\pi : L \to X$

Then $\pi(t \cdot_L v) = \pi(\chi_m(t) \cdot t \cdot_L v)$ ← $\chi_m(t) \in \mathbb{G}_m \hookrightarrow T$ linearisation $\Rightarrow \pi$ T-eq.

$= \chi_m(t)\, \pi(t \cdot_L v)$ ← $\chi_m(t) \in \mathbb{G}_m$ and X projective.

$= \pi(t \cdot_L v)$

Thm

1) Chamber decomposition.

The set of $\mathbb{Q}$-linearisation of L

$$Amp^G_{\mathbb{Q}}(X) := \{ \text{Ample } \mathbb{Q}\text{-line bundle } L \text{ with a } G \curvearrowright L \}$$

i.e. L which remembers the $G \curvearrowright L$

· Q: What does it mean for $G \curvearrowright (X, H)$ where H $\mathbb{Q}$-line bundle on X

$\exists\, m \in \mathbb{Z}_{>0}$ s.t. $H^{\otimes m}$ is a genuine line bundle.

$G \curvearrowright (X, H^{\otimes m})$

is divided onto finitely many polyhedral chambers.

2) Constancy within chambers

Let c, c' be in $Amp^G_{\mathbb{Q}}(X)$

If c, c' lie in the same locally closed chamber, then $Y_c = Y_{c'}$.

3) Morphisms across Walls.

If c specialization of c' (i.e. $c \in \overline{\{c'\}}$)
then $\exists$ a proper contraction

$$\pi : Y_{c'} \longrightarrow Y_c$$

with $\pi_* \mathcal{O}_{Y_{c'}} = \mathcal{Y}_c$

What we need from VAIT for weighted hyperplane arrangement is:

If $\begin{cases} \cdot (X, L) \text{ polarized variety} \\ \cdot\ L \text{ ample linearisation} \\ \cdot\ Z \subset X \text{ closed } G\text{-invariant subvariety} \end{cases}$

then $Z /\!\!/_{L|_Z} G$ is a closed subvariety of $X /\!\!/_L G$.