

The (Direct) Comparison Test (for series)

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

- If $\sum b_n$ converges and $a_n \leq b_n$ for all n , then $\sum a_n$ also converges.
- If $\sum b_n$ diverges and $a_n \geq b_n$ for all n , then $\sum a_n$ also diverges.

To use: Given $\sum a_n$, you pick $\sum b_n$.

Note: You can ONLY use Comparison Test on series with positive terms.

Make sure you check that $a_n, b_n \geq 0$!

Good series to compare to:

- geometric series

$\sum ar^n$ cvgs if $|r| < 1$, diverges otherwise

- p-series

$\sum \frac{1}{n^p}$ cvgs if $p > 1$, diverges if $p \leq 1$

DETAILED SOLUTION

ex: $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$

$$a_n = \frac{1}{2^n + n} \geq 0 \quad \text{for all } n$$

Compare to $b_n = \frac{1}{2^n}$. Then $b_n \geq 0$ for all n .

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} \quad \text{converges because its geometric with } r = \frac{1}{2} \text{ so } |r| < 1.$$

Want: $a_n \leq b_n$ for all n

$$\frac{1}{2^n + n} \leq \frac{1}{2^n}$$

$$2^n \leq 2^n + n$$

↖ This is true because n is a positive integer.

Conclusion:

$$\sum_{n=1}^{\infty} \frac{1}{2^n + n} \quad \text{converges by the Comparison Test.}$$

BARE MINIMUM SOLUTION:

ex: $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$ $\leftarrow$ positive terms $\checkmark$

Pick $b_n = \frac{1}{2^n} \geq 0 \checkmark$

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ geometric, $|r| = |\frac{1}{2}| < 1$ converges

$$\frac{1}{2^n + n} \leq \frac{1}{2^n}$$

$\nearrow$
larger denominator
so smaller

original series converges by Comp. Test

BAD SOLUTION (will lose points for write-up)

ex: $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$

$b_n = \frac{1}{2^n}$ converges, $r = \frac{1}{2} < 1$

$a_n \leq b_n$ and $\frac{1}{2^n}$ converges $\leftarrow$ see note below

$\frac{1}{2^n + n}$ converges by Comp. Test

Standard of Proof: can another Calc 2 student read and understand your solution?

Notes:

- Make sure to include the " $\sum$ " when talking about the SERIES not the sequence.

In the last solution,

$\frac{1}{2^n}$ should be $\sum_{n=1}^{\infty} \frac{1}{2^n}$

- Compare the terms not the series

$a_n \leq b_n$ NOT $\sum a_n \leq \sum b_n$

ex: $\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$

← positive terms ✓

Pick $b_n = \frac{1}{n^2}$ (drop lower order terms)

Then $b_n \geq 0$ for all n . ✓

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges because } p=2 > 1.$$

Want: $\frac{1}{n^2+n+1} \leq \frac{1}{n^2}$ ← true ✓

↑
larger denominator

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n+1} \text{ converges by Comparison Test}$$

ex: $\sum_{n=1}^{\infty} \frac{\cos^2 n}{\sqrt{n^3+n}}$ $\leftarrow$ positive terms $\checkmark$

Compare to $b_n = \frac{1}{\sqrt{n^3}} \geq 0 \checkmark$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

$$p = \frac{3}{2} > 1 \text{ so converges}$$

Want: $\frac{\cos^2 n}{\sqrt{n^3+n}} \stackrel{?}{\leq} \frac{1}{\sqrt{n^3}}$ Tip: compare numerator and denominator separately.

$\uparrow$
true because $\cos^2 n \leq 1$
and $\sqrt{n^3+n} \geq \sqrt{n^3}$

$\sum a_n$ converges by Comparison Test.

ex: $\sum_{n=2}^{\infty} \frac{\sqrt{n^4+1}}{n^3-2}$

← positive terms ✓

(notice we start at $n=2$)

Pick $b_n = \frac{\sqrt{n^4}}{n^3} = \frac{n^2}{n^3} = \frac{1}{n} \geq 0$ ✓

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series)

Want: $\frac{1}{n} \stackrel{?}{\leq} \frac{\sqrt{n^4+1}}{n^3-2}$

Tip: use the form
of b_n from
before we simplified

$$\frac{\sqrt{n^4}}{n^3} \leq \frac{\sqrt{n^4+1}}{n^3-2}$$

$$b_n = \frac{\sqrt{n^4}}{n^3}$$

↑ true because $\sqrt{n^4} \leq \sqrt{n^4+1}$

and $n^3 \geq n^3-2$

Limit Comparison Test (LCT)

If $\sum a_n$ and $\sum b_n$ are both series with positive terms and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c \quad \text{where} \quad 0 < c < \infty$$

strict inequalities!

then either both series converge or both diverge.

Note: If we get $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ or ∞ , the test is inconclusive $\Rightarrow$ need to try a different test OR pick a different b_n

Intuition: Remember that we can think of $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ as a race between a_n and b_n .

Assuming that $a_n, b_n \rightarrow 0$,

if $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$, $\{a_n\}$ went to 0 faster

if $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$, $\{b_n\}$ went to 0 faster

if $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$ for some $0 < c < \infty$,
then they grow at essentially the same rate
(proportional to each other)

LCT says that if $\{a_n\}$ and $\{b_n\}$ grow at the same rate, $\sum a_n$ and $\sum b_n$ have the same convergence behavior.

ex: $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ $\leftarrow$ positive terms $\checkmark$

Pick $b_n = \frac{1}{n} \geq 0$ $\checkmark$ ($\sqrt{n}$ is a lower order term)

Then $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, harmonic

(If using DCT, we would want $\frac{1}{n} \leq \frac{1}{n+\sqrt{n}}$ but this isn't true...)

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n+\sqrt{n}}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{n+\sqrt{n}} \quad \leftarrow \frac{\infty}{\infty} \\ &\stackrel{LH}{=} \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{2}n^{-1/2}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{2\sqrt{n}}} = 1 \end{aligned}$$

$\downarrow$
0

Since $0 < 1 < \infty$, $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ also diverges by LCT.

ex: $\sum_{n=1}^{\infty} \frac{\arctan(n)}{\sqrt{n}}$ $\leftarrow$ positive terms $\checkmark$

Pick $b_n = \frac{1}{\sqrt{n}} \geq 0 \checkmark$

Then $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ $p = \frac{1}{2} < 1$
diverges

LCT: $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{\arctan(n)}{\sqrt{n}} \right)}{\left(\frac{1}{\sqrt{n}} \right)}$
 $= \lim_{n \rightarrow \infty} \arctan(n) = \frac{\pi}{2}$

Since $0 < \frac{\pi}{2} < \infty$, the original series also diverges by Limit Comparison Test.

Warning for using LCT:

When choosing b_n you need to match the speed at which a_n goes to 0, you can't over/undershoot like you can with DCT

$$\text{E.g. } \sum_{n=1}^{\infty} \frac{1}{e^n + n^2}$$

If using DCT, you could use

$$b_n = \frac{1}{e^n}$$

OR

$$b_n = \frac{1}{n^2}$$

$$\frac{1}{e^n + n^2} \leq \frac{1}{e^n} \checkmark$$

$$\frac{1}{e^n + n^2} \leq \frac{1}{n^2} \checkmark$$

But if using LCT, only $b_n = \frac{1}{e^n}$ works

Using $b_n = \frac{1}{n^2}$, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{b_n}{a_n} &= \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{e^n + n^2}\right)} = \lim_{n \rightarrow \infty} \frac{e^n + n^2}{n^2} \\ &= \lim_{n \rightarrow \infty} \left(\frac{e^n}{n^2} + 1\right) = \infty \end{aligned}$$

↑
inconclusive
(LCT fails w/ this b_n)

This is because for $a_n = \frac{1}{e^n + n^2}$, e^n takes over as n gets large

↑
dominant term

Also, when choosing b_n for LCT, constants matter less than they do for DCT.

ex:
$$\sum_{n=1}^{\infty} \frac{3n-2}{2n^3+5}$$

If using DCT, pick $b_n = \frac{3n}{2n^3} = \frac{3}{2n^2}$, NOT $\frac{1}{n^2}$

For LCT, $b_n = \frac{1}{n^2}$ will work just fine.

ex:
$$\sum_{n=1}^{\infty} \frac{\arctan(n)}{2^n}$$

For DCT, need to pick $b_n = \frac{\pi/2}{2^n}$.

For LCT, can use $b_n = \frac{1}{2^n}$.