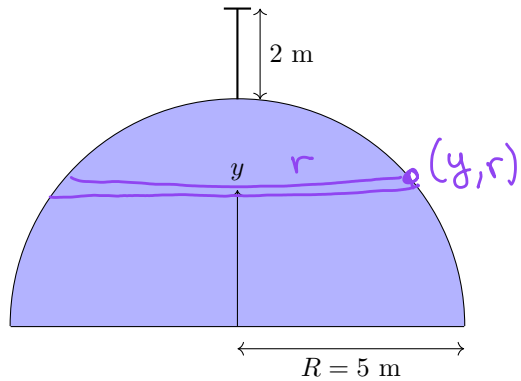
Exam 2- Practice

(i) **Work**

(1) A hemispherical tank of radius 5 meters is full of water. The tank is positioned with its flat face at the bottom. Find the work required to pump all the water out through a pipe that extends 2 meters above the top of the tank. (Use $\rho = 1000$ kg/m^3 for water density)



$y^2 + r^2 = 5^2$

$r = \sqrt{25 - y^2}$

$V_s = \pi \cdot r^2 \cdot \Delta y$

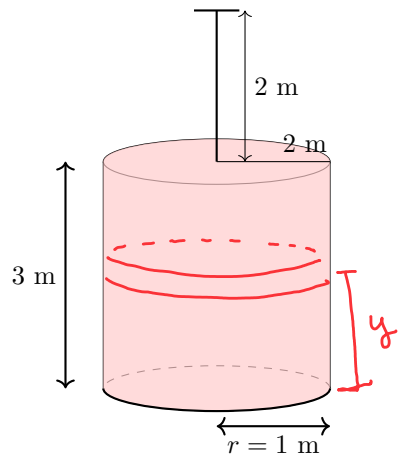
$= \pi \cdot (\sqrt{25 - y^2})^2 \, \Delta y$

$m_s = \rho \pi (25 - y^2) \, \Delta y$

$w_s = g \rho \pi (25 - y^2) \, \Delta y$

$W = \int_0^5 g \rho \pi (25 - y^2)(7 - y) \, dy$

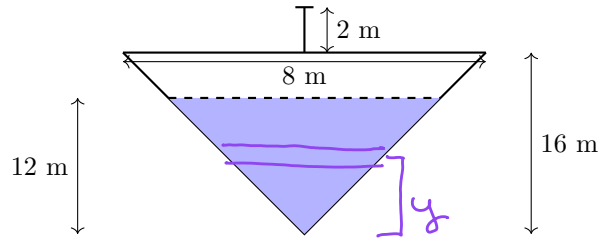
(2) A vertical cylindrical tank with radius 1 meters and height 3 meters contains a liquid whose density varies with depth according to $\rho(y) = 8 + 5y$ kg/m^3, where y is measured from the bottom. Find the work needed to pump all the liquid to a level 2 meters above the top of the tank.

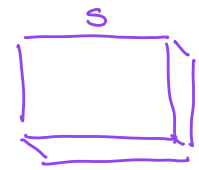


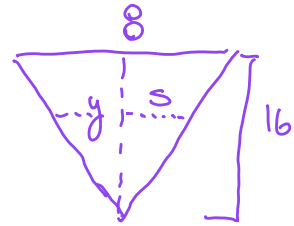
$$V_S = \pi \cdot r^2 \cdot \Delta y$$
$$= \pi \cdot 1^2 \, \Delta y$$
$$m_S = \rho \, \pi \cdot 1^2 \, \Delta y$$
$$= (8+5y)\pi \quad \Delta y$$
$$w_S = (8+5y) \cdot g \pi \, \Delta y$$

$$W = \int_0^3 (8+5y) \, g \pi \, (5-y) \, dy$$

(3) Consider an inverted pyramidal tank containing liquid with density ρ kg/m^3. The tank has a square cross-section with 8-meter sides at the top and comes to a point at the bottom, with a total height of 16 meters. If the tank currently contains liquid up to a depth of 12 meters, set up an integral expression that would calculate the work needed to pump all the liquid out through an outlet located 2 meters above the top of the tank. Do not evaluate the integral.







$$\frac{s}{y} = \frac{8}{16}$$

$$s = \frac{y}{2}$$

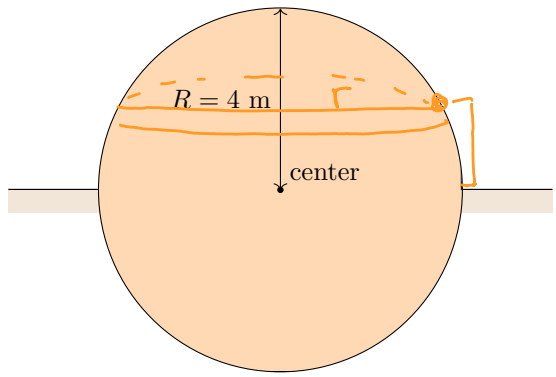
$$V_s = \left(\frac{y}{2}\right)^2 \cdot \Delta y$$

$$m_s = \rho \left(\frac{y}{2}\right)^2 \Delta y$$

$$w_s = \rho \cdot g \left(\frac{y}{2}\right)^2 \Delta y$$

$$W = \int_0^{12} \rho \cdot g \left(\frac{y}{2}\right)^2 (18 - y)\, dy$$

(4) A spherical tank of radius 4 meters is buried in the ground with its center at ground level. The tank is full of gasoline ($\rho = 720$ kg/m^3). Find the work required to pump all the gasoline to a height of 5 meters above ground level.



$r^2 + y^2 = 4^2$

$r = \sqrt{16 - y^2}$

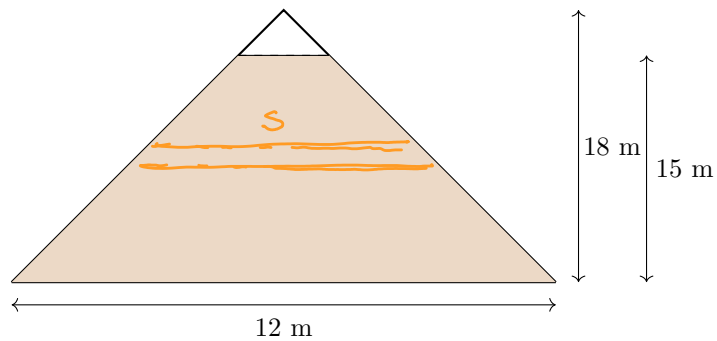
$V_S = \cdot (\sqrt{16-y^2})^2 \cdot \Delta y$

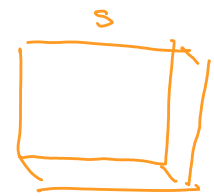
$m_S = \rho \, (\sqrt{16-y^2})^2 \, \Delta y$

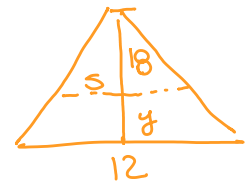
$w_S = g\rho \, (16 - y^2) \, \Delta y$

$W = \int_{-4}^{4} g\rho \cdot (16 - y^2) \, dy.$

(5) A pyramidal storage tank sits on its base and tapers to a point at the top. The base is a square measuring 12 meters on each side, and the tank reaches a height of 18 meters. The tank holds liquid (with density ρ kg/m^3) filled to a level of 15 meters. Express as an integral the total work that would be required to remove all the liquid by pumping it out at the top of the tank.







$$\frac{s}{18-y} = \frac{12}{18} = \frac{2}{3}$$

$$V_s = s^2 \cdot \Delta y$$
$$= \left(\tfrac{2}{3}(18-y)\right)^2 \Delta y$$

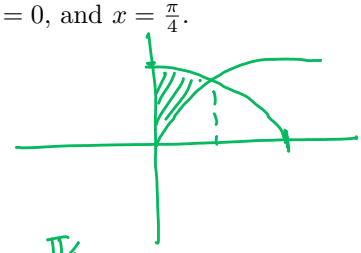
$$m_s = \rho \left(\tfrac{2}{3}(18-y)\right)^2 \Delta y$$
$$w_s = g\rho \left(\tfrac{2}{3}(18-y)\right)^2 \Delta y$$

$$W = \int_0^{15} g\rho \left(\tfrac{2}{3}(18-y)\right)^2 \cdot (18-y)\, dy$$

(ii) **Center of mass**

(1) Find the centroid of the region bounded by $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \frac{\pi}{4}$.

$$\bar{x} = \frac{\int_0^{\pi/4} x(\cos x - \sin x)\,dx}{A} = \frac{[\sqrt{2}-2]}{2(\sqrt{2}-1)}$$



$$\bar{y} = \frac{\int_0^{\pi/4} \frac{1}{2}(\cos^2 x - \sin^2 x)\,dx}{A}$$

$$= \frac{1}{4(\sqrt{2}-1)}$$

$$A = \int_0^{\pi/4} \cos x - \sin x\,dx$$

$$= \sin x + \cos x \Big|_0^{\pi/4} = \sqrt{2} - 1$$

(2) Find the arc length of the curve $y = \frac{1}{3}(x^2+2)^{3/2}$ from $x = 0$ to $x = 3$.

$$L = \int_0^3 \sqrt{1^2 + (t(t^2+2)^{1/2})^2}\,dt$$

$$= \frac{x^3}{3} + x \Big|_0^3 = 12$$

$$\begin{cases} y = \frac{1}{3}(t^2+2)^{3/2} \\ x = t \end{cases}$$

$$\frac{dy}{dt} = \cancel{\frac{1}{2}}(t^2+2)^{1/2} \cdot (\cancel{2}t)$$

(3) Find the arc length of the curve $y = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2}$ from $x = 1$ to $x = 9$.

$$L = \int_1^9 \sqrt{1 + \left(t^{1/2} - \frac{t^{-1/2}}{4}\right)^2}\,dt$$

$$= \frac{56}{3}$$

$$\begin{cases} x = t \\ y = \frac{2}{3}t^{3/2} - \frac{1}{2}t^{1/2} \end{cases}$$

$1 \le t \le 9$

$$\frac{dy}{dt} = t^{1/2} - \frac{t^{-1/2}}{4}$$

(4) Find the arc length of the curve $y = \frac{2}{3}(x-1)^{3/2}$ from $x = 1$ to $x = 4$.

$$L = \int_1^4 \sqrt{1 + (t-1)}\,dt = \frac{14}{3}$$

$$\begin{cases} x = t \\ y = \frac{2}{3}(t-1)^{3/2} \end{cases}$$

$$\frac{dy}{dt} = (t-1)^{1/2}$$

$1 \le t \le 4$

(5) Find the arc length of the curve $y = \frac{1}{2}(e^x + e^{-x})$ from $x = 0$ to $x = 2$.

$$L = \int_0^2 \sqrt{1 + \left(\frac{1}{2}(e^t - e^{-t})\right)^2}\, dt$$

$$= \frac{1}{2}(e^2 - e^{-2})$$

$$\begin{cases} x = t \\ y = \frac{1}{2}(e^t + e^{-t}) \end{cases}$$

$$0 \le t \le 2$$

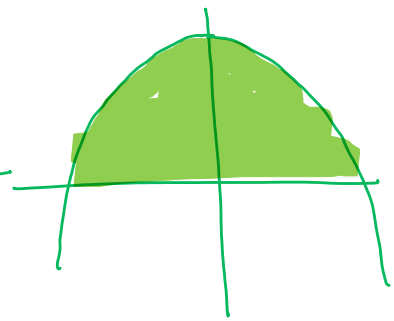
$$\frac{dy}{dt} = \frac{1}{2}(e^t - e^{-t})$$

(6) Find the centroid of the region bounded by $y = 4 - x^2$ and the x-axis.

$$A = \int_{-2}^{2} (4 - x^2)\, dx = \frac{32}{3}$$

$$\bar{x} = \frac{\int_{-2}^{2} x(4 - x^2)\, dx}{A} = 0 \rightarrow \text{by symmetry}$$

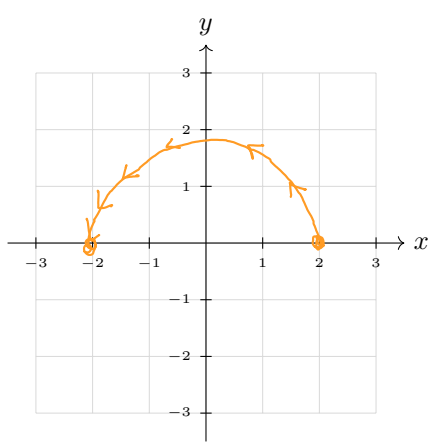
$$\bar{y} = \frac{\int_{-2}^{2} (4 - x^2)^2\, dx}{2A} = \frac{8}{5}$$



(iii) **Parametric equations**

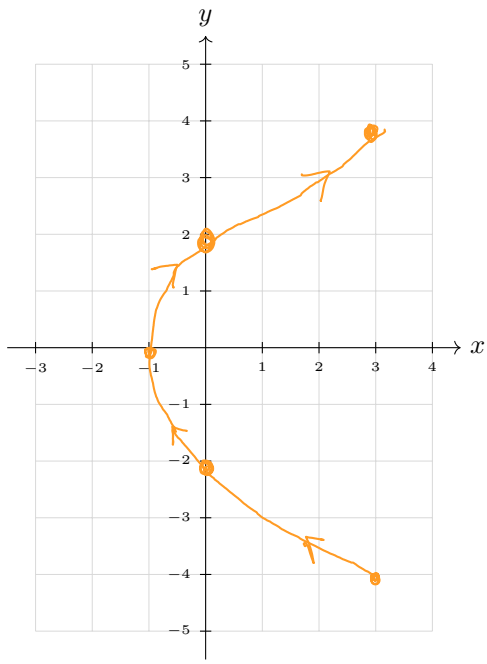
(1) Draw the parametrized curve given by

$$\begin{cases} x = 2\sin t \\ y = 2\cos t \end{cases} \qquad 0 \le t \le \pi.$$



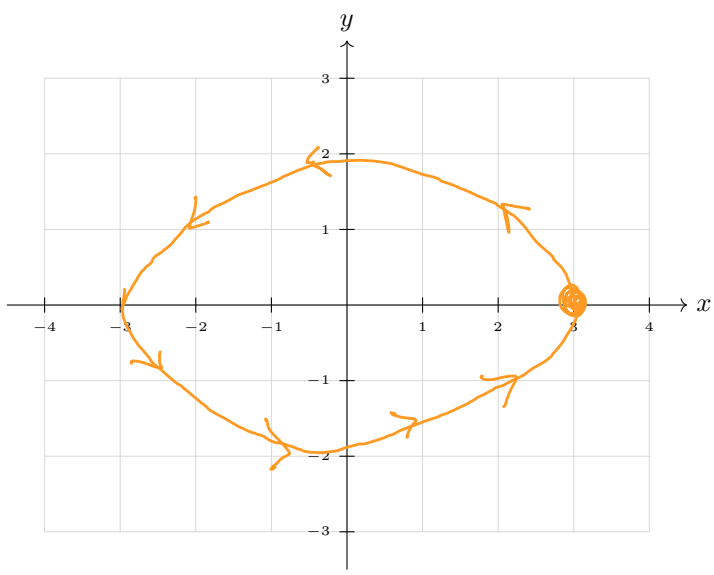
(2) Draw the parametrized curve given by

$$\begin{cases} x = t^2 - 1 \\ y = 2t \end{cases} \qquad -2 \le t \le 2.$$



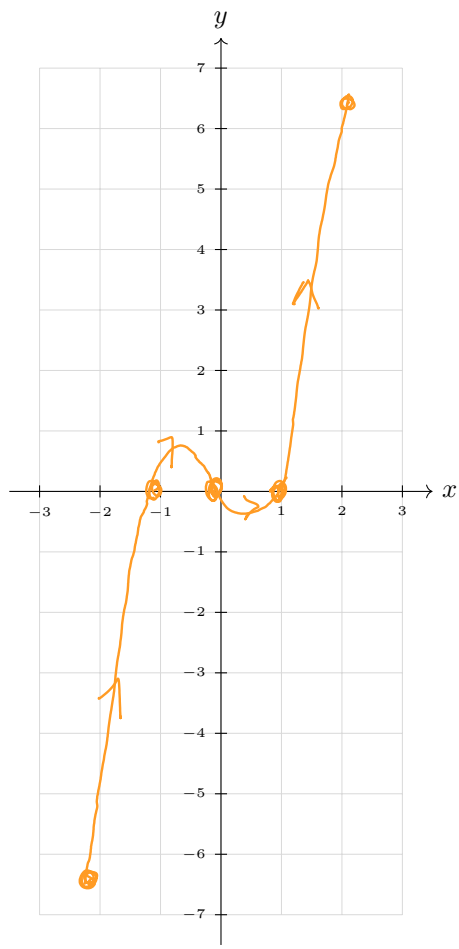
(3) Draw the parametrized curve given by

$$\begin{cases} x = 3\cos t \\ y = 2\sin t \end{cases} \qquad 0 \le t \le 2\pi.$$



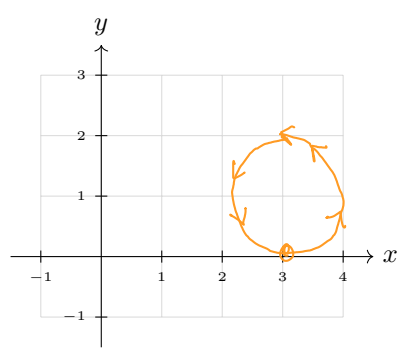
(4) Draw the parametrized curve given by

$$\begin{cases} x = t \\ y = t^3 - t \end{cases} \qquad -2 \le t \le 2.$$



(5) Draw the parametrized curve given by

$$\begin{cases} x = 2 + \cos t \\ y = 1 + \sin t \end{cases} \qquad 0 \le t \le 2\pi.$$



(iv) **Tangents of parametric Curves**

(1) Consider the parametric equations:

$$x = t^2 - 4$$
$$y = t^3 - 3t$$

a) Find $\frac{dy}{dx}$ in terms of t.
b) Find the equation of the tangent line at $t = 2$.
c) Find all points where the tangent line has slope 0.

a) $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2-3}{2t}$

b) At $t=2 \rightarrow x=0$ & $y=2$ $\frac{dy}{dx} = \frac{9}{4}$

c) Tangent line $y-2 = \frac{9}{4}(x-0)$

(2) For the curve defined by:

$$x = 3\cos(t)$$
$$y = 2\sin(t)$$
$$0 \le t \le 2\pi$$

a) Find all points where the curve has a horizontal tangent line.
b) Find all points where the curve has a vertical tangent line.
c) Write the equation of the tangent line at $t = \pi/4$.

a) $\frac{dy}{dt} = 0$ & $\frac{dx}{dt} \neq 0 \Rightarrow 2\cos t = 0$ and $\frac{dx}{dt} = -3\sin(t)$

$t = \frac{\pi}{2}, \frac{3\pi}{2}$

$\boxed{\frac{dx}{dt}\left(\frac{\pi}{2}\right) \neq 0 \neq \frac{dx}{dt}\left(\frac{3\pi}{2}\right)}$

then $t = \frac{\pi}{2} \rightarrow (0,2)$

$t = \frac{3\pi}{2} \rightarrow (0,-2)$

b) $\frac{dx}{dt} = 0$ & $\frac{dy}{dt} \neq 0$

$-3\sin t = 0 \rightarrow t = 0, \pi$ $\frac{dy}{dt}(0) \neq 0 \neq \frac{dy}{dt}(\pi)$ by a.

$t=0 \rightarrow (3,0)$ & $t = \pi \rightarrow (-3,0)$

c) $t = \frac{\pi}{4}$ $x = \frac{3}{\sqrt{2}}$ $y = \sqrt{2}$ $\frac{dy}{dx} = \frac{-2}{3} \Rightarrow y - \sqrt{2} = \left(\frac{-2}{3}\right)\left(x - \frac{3}{\sqrt{2}}\right)$

(3) For the curve defined by:

$$x = t^3 + 4t + 5$$
$$y = t^2 + 6t$$
$$0 \leq t$$

a) Find all points where the curve has a horizontal tangent line (If they exist).
b) Find all points where the curve has a vertical tangent line (If they exist).
c) Write the equation of the tangent line at $t = 0$.

a) $\frac{dy}{dt} = 2t + 6 = 0 \Rightarrow t = -3$ and $\frac{dx}{dt}(-3) \neq 0$
Hence, horizontal tangent line at $(-34, -9)$

b) $\frac{dx}{dt} = 3t^2 + 4 \neq 0$, No vertical tangent lines

c) $\underline{t=0}$ $x = 5$, $y = 0$ $\frac{dy}{dx} = \frac{6}{4} = \frac{3}{2}$

$y - 0 = \frac{3}{2}(x - 5)$

(4) The curve defined by:

$$x = t^2 - 1$$
$$y = t^3 - t$$

crosses itself at the point $(0, 0)$.

a) Find the two values of t that give the point $(0, 0)$.
b) Find the slopes of both tangent lines at this point.

a) $t^2 - 1 = 0$ & $t^3 - t = 0 \Rightarrow t = \pm 1$
$(t-1)(t+1) = 0$ $t(t-1)(t+1) = 0$

b) $\frac{dy}{dx} = \frac{3t^2 - 1}{2t}$

at $t = 1 \rightarrow \frac{dy}{dx} = 1$

at $t = -1 \rightarrow \frac{dy}{dx} = -1$

(5) For the logarithmic spiral:

$$x = e^t \cos(t)$$
$$y = e^t \sin(t)$$
$$0 \le t \le 2\pi$$

a) Show that $\frac{dy}{dx} = \frac{\sin(t)+\cos(t)}{\cos(t)-\sin(t)}$.

b) Find all values of t where the tangent line passes through the origin.

a) $\frac{dx}{dt} = e^t \cos t - e^t \sin t$ $\quad \frac{dy}{dt} = e^t \sin t + e^t \cos t$

$\frac{dy}{dx} = \frac{\cancel{e^t}(\cos t + \sin t)}{\cancel{e^t}(\cos t - \sin t)} = \frac{\cos t + \sin t}{\cos t - \sin t}$

b) Ignore it!

(v) **Arc Lenght**

(1) Write the integral that represents the arc length of the curve $y = x^2$ from $x = 0$ to $x = 2$.

$L = \int_0^2 \sqrt{(1)^2 + (2t)^2}\, dt$

$= \int_0^2 \sqrt{1 + 4t^2}\, dt$

$\begin{cases} x = t & 0 \le t \le 2 \\ y = t^2 \end{cases}$

$\frac{dx}{dt} = 1 \quad \frac{dy}{dt} = 2t$

(2) Write the integral that represents the arc length of the curve $y = \ln(x)$ from $x = 1$ to $x = e$.

$L = \int_1^e \sqrt{(1)^2 + \left(\frac{1}{t}\right)^2}\, dt$

$= \int_1^e \sqrt{1 + \frac{1}{t^2}}\, dt$

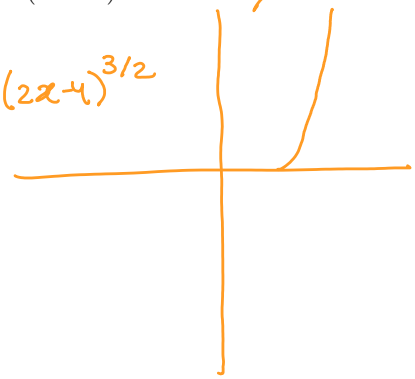
$\begin{cases} x = t \\ y = \ln(x) \end{cases} \quad 1 \le t \le e$

$\frac{dx}{dt} = 1 \quad \frac{dy}{dt} = \frac{1}{\ }$

(3) Write the integral that represents the arc length of the curve $y = (2x-4)^{5/2}$ from $x = \cancel{1}\ 2$ to $x = 5$.

$$L = \int_2^5 \sqrt{1 + 25(2x-4)^3}\, dx$$

$$\frac{dy}{dt} = 5(2x-4)^{3/2}$$



(4) Write the integral that represents the arc length of the parametric curve

$$\begin{cases} x = t^3 \\ y = t^2 \end{cases}$$

from $t = 0$ to $t = 2$.

$$L = \int_0^2 \sqrt{(3t^2)^2 + (2t)^2}\, dt$$

$$= \int_0^2 \sqrt{9t^4 + 4t^2}\, dt$$

$$\frac{dx}{dt} = 3t^2$$

$$\frac{dy}{dt} = 2t$$

(5) Write the integral that represents the arc length of the curve $x = \sqrt{y}$ from $y = 1$ to $y = 4$.

$$L = \int_1^4 \sqrt{\left(\frac{1}{2\sqrt{t}}\right)^2 + 1^2}\, dt$$

$$= \int_1^4 \sqrt{\frac{1}{4t} + 1}\, dt$$

$$\begin{cases} x = \sqrt{t} \\ y = t \end{cases} \quad 1 \le t \le 4$$

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}}$$

$$\frac{dy}{dt} = 1$$

(vi) **Polar Coodinates**

(1) Convert the point with polar coordinates $(r, \theta) = \left(4, \frac{\pi}{3}\right)$ to Cartesian coordinates.

$$x = 4\cos\left(\frac{\pi}{3}\right) = 2$$

$$y = 4\sin\left(\frac{\pi}{3}\right) = 2\sqrt{3}$$

$$(x, y) = (2, 2\sqrt{3})$$

(2) Convert the point with polar coordinates $(r, \theta) = \left(3, \frac{3\pi}{4}\right)$ to Cartesian coordinates.

$$x = 3 \cdot \cos\left(\frac{3\pi}{4}\right) = -\frac{3\sqrt{2}}{2}$$

$$y = 3\sin\left(\frac{3\pi}{4}\right) = \frac{3\sqrt{2}}{2}$$

$$(x, y) = \left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$$

(3) Convert the point with polar coordinates $(r, \theta) = \left(5, -\frac{\pi}{6}\right)$ to Cartesian coordinates.

$$x = 5\cos\left(-\frac{\pi}{6}\right) = \frac{5\sqrt{3}}{2}$$

$$y = 5\sin\left(-\frac{\pi}{6}\right) \quad \frac{-5}{2}$$

$$(x, y) = \left(\frac{5\sqrt{3}}{2}, \frac{-5}{2}\right)$$

(4) Convert the point with Cartesian coordinates $(x, y) = (3, 3)$ to polar coordinates.

$$r = \sqrt{3^2 + 3^2} = \sqrt{18} = \sqrt{2} \cdot 3$$

$$\theta = \arctan\left(\frac{3}{3}\right) = \frac{\pi}{4}$$

$$(r, \theta) = \left(\sqrt{2} \cdot 3, \frac{\pi}{4}\right)$$

(5) Convert the point with Cartesian coordinates $(x, y) = (0, -5)$ to polar coordinates.

$r = \sqrt{0^2 + 5^2} = 5$

$\theta = \frac{3\pi}{2}$

$(r, \theta) = \left(5, \frac{3\pi}{2}\right)$

(6) Convert the point with Cartesian coordinates $(x, y) = (-4, -4)$ to polar coordinates.

$r = \sqrt{(-4)^2 + (-4)^2} = \sqrt{2} \cdot 2$

$\theta = \arctan\left(\frac{-4}{-4}\right) = \frac{5\pi}{4}$ → because x & y are negative

$(r, \theta) = \left(\sqrt{2} \cdot 2, \frac{5\pi}{4}\right)$

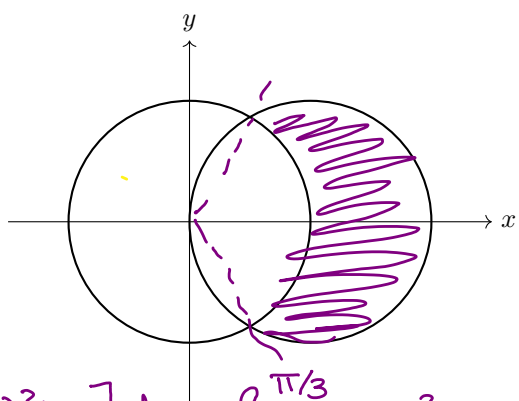
(vii) **Areas in polar coordinates**

(1) Write an integral to compute the area of the region inside $r = 4\cos\theta$ and outside $r = 2$.

Intersection:

$4\cos\theta = 2$

$\theta = \pm \frac{\pi}{3}$



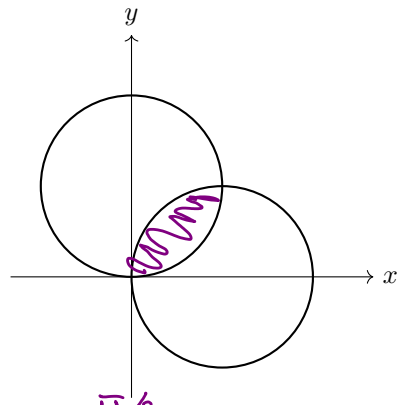
$$A = \int_{-\pi/3}^{\pi/3} \frac{1}{2}\left[(4\cos\theta)^2 - 4\right] d\theta = \int_0^{\pi/3} (16\cos^2\theta - 4)\, d\theta$$

(2) Write an integral to compute the area of the region inside both $r = 3\sin\theta$ and $r = 3\cos\theta$.

Intersection:

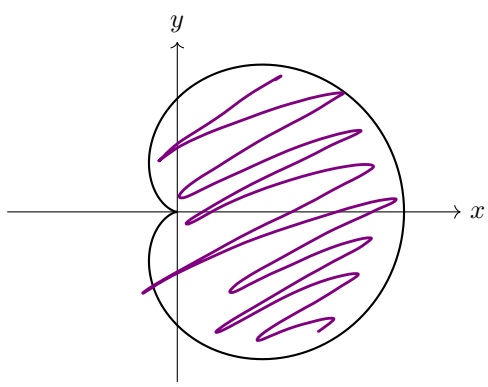
$\sin\theta = \cos\theta$

$\theta = \frac{\pi}{4}$



$$A = \int_0^{\pi/4} \frac{1}{2}(3\cos\theta)^2\,d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}(3\sin\theta)^2\,d\theta$$

(3) Write an integral to compute the area of the region inside $r = 1 + \cos\theta$ (cardioid).



$$A = \int_0^{2\pi} \frac{1}{2}(1+\cos\theta)^2\,d\theta$$

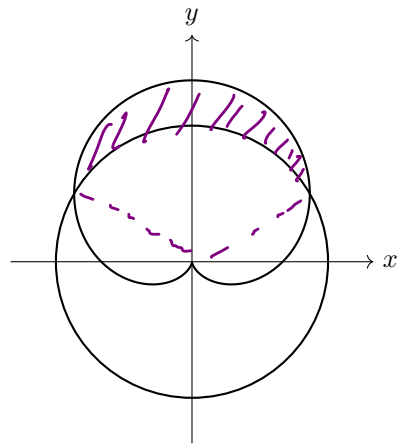
(4) Write an integral to compute the area of the region inside $r = 2 + 2\sin\theta$ and outside $r = 3$.

Intersection

$2 + 2\sin\theta = 3$

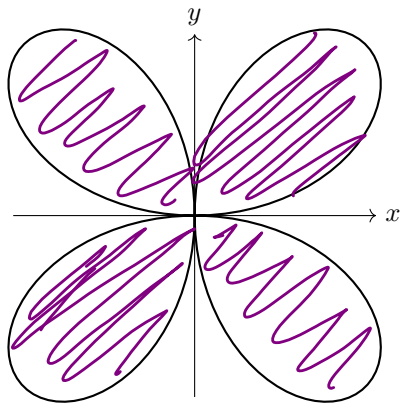
$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$



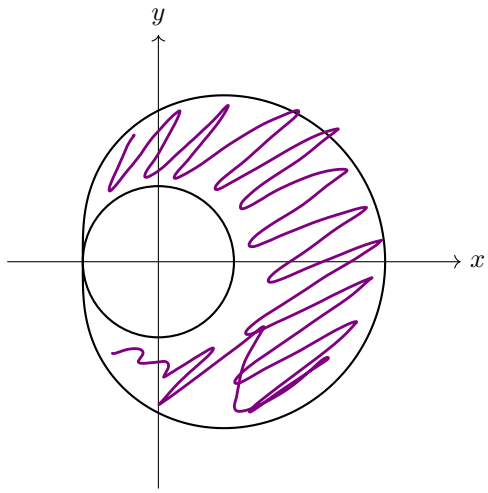
$$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}\left[(2+2\sin\theta)^2 - 9\right] d\theta$$

(5) Write an integral to compute the area of the region inside $r = 4\sin(2\theta)$ (four-petaled rose).



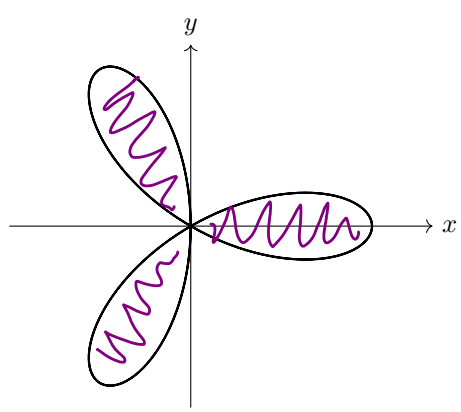
$$A = 4\int_0^{\pi/2} \frac{1}{2}\left(4\sin(2\theta)\right)^2 d\theta$$

(6) Write an integral to compute the area of the region inside $r = 2 + \cos\theta$ and outside $r = 1$.



$$A = \int_0^{2\pi} \frac{1}{2}\left[(2+\cos\theta)^2 - 1\right] d\theta$$

(7) Write an integral to compute the area of the region inside $r = 3\cos(3\theta)$ (three-petaled rose).



$$A = 3\int_{\frac{\pi}{2}}^{\pi} \left(\frac{1}{2}\right)\left(3\cos(3\theta)\right)^2 d\theta$$

(8) Write an integral to compute the area of the region inside $r = 2\cos(2\theta)$ (lemniscate) and outside $r = 1$.

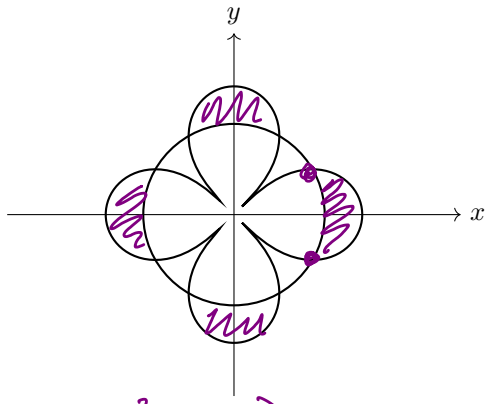
Intersection

$1 = 2\cos(2\theta)$

$\frac{1}{2} = \cos(2\theta)$

$2\theta = \frac{\pi}{3}, -\frac{\pi}{3}$

$\theta = \frac{\pi}{6}, -\frac{\pi}{6}$



$$A = 4\int_{-\frac{\pi}{6}}^{\pi/6} \frac{1}{2}\left((2\cos(2\theta))^2 - 1^2\right) d\theta$$

(viii) **Arc Lenght in polar coordinates**

(1) Find the arc length of the circle $r = 4$ from $\theta = 0$ to $\theta = \pi/2$.

$$L = \int_0^{2\pi} 4\, d\theta = 2\pi$$

(2) Find the total arc length of the cardioid $r = 2(1 + \cos\theta)$ for $\theta \in [0, 2\pi]$.

$$L = \int_0^{2\pi} \sqrt{4(1+\cos\theta)^2 + 4\sin^2\theta}\, d\theta = 16$$

$$\frac{dr}{d\theta} = -2\sin\theta$$

(3) Calculate the arc length of the spiral $r = \theta$ from $\theta = 0$ to $\theta = 2\pi$.

$$L = \int_0^{2\pi} \sqrt{\theta^2 + 1}\, d\theta$$

(4) Find the arc length of one petal of the four-petaled rose $r = 3\sin(2\theta)$ from $\theta = 0$ to $\theta = \pi/2$.

$$L = \int_0^{\pi/2} \sqrt{9\sin^2(2\theta) + 36\cos^2(2\theta))}\, d\theta$$

(5) Determine the arc length of the logarithmic spiral $r = e^{\theta/2}$ from $\theta = 0$ to $\theta = \pi$.

$$L = \int_0^{\pi} \sqrt{e^{\theta} + e^{\theta}/4}\, d\theta$$

(ix) **Differential equations**

(1) Check that $y = x^2 + \frac{1}{x}$ is a solution to the differential equation:

$$xy' + y = 3x^2$$

Also verify that it satisfies the initial condition $y(1) = 2$.

$y' = 2x - \frac{1}{x^2}$

$x(2x - \frac{1}{x^2}) + (x^2 + \frac{1}{x}) = 2x^2 - \cancel{\frac{x}{x^2}} + x^2 + \cancel{\frac{1}{x}} = 3x^2$ ✓

$y = 1^2 + \frac{1}{1} = 2$ ✓

(2) Show that $y = \tan(2x^2)$ is a solution to the differential equation:

$$\frac{dy}{dx} = 4x(1 + y^2)$$

and verify that it satisfies the initial condition $y(0) = 0$.

$\frac{dy}{dx} = \sec^2(2x^2)\ 4x$

$4x(1+y^2) = 4x(1 - \tan^2(2x^2))$

$= 4x \sec^2(2x^2)$

$= \frac{dy}{dx}$ ✓

$y = \tan(2 \cdot 0^2) = 0$ ✓

(3) Verify that $x = t^2 - 2t + 3$ is a solution to the initial value problem:

$$\frac{d^2x}{dt^2} - 2 = 0$$
$$x(0) = 3$$
$$x'(0) = -2$$

$$\frac{dx}{dt} = 2t - 2$$

$$\frac{d^2x}{dt^2} = 2 \longrightarrow \frac{d^2x}{dt^2} - 2 = 0 \checkmark$$

$$x(0) = 0^2 - 2\cdot 0 + 3 = 3 \checkmark$$

$$x'(0) = 2\cdot 0 - 2 = -2 \checkmark$$

(4) Solve the differential equation:

$$\frac{dy}{dx} = \frac{y^2}{x^2}$$

with the initial condition $y(1) = 2$.

$$\int \frac{dy}{y^2} = \int \frac{dx}{x^2}$$

$$-\frac{1}{y} = -\frac{1}{x} + C$$

$$-\frac{1}{2} = -\frac{1}{1} + C$$

$$\frac{1}{2} = C$$

$$-\frac{1}{y} = -\frac{1}{x} + \frac{1}{2}$$

$$y = \left(\frac{1}{x} - \frac{1}{2}\right)^{-1}$$

(5) Find the general solution to:

$$(1 + e^x)y\,dy = e^x\,dx$$

$$\int y\,dy = \int \frac{e^x}{1+e^x}\,dx$$

$$\frac{y^2}{2} = \ln(e^x+1) + C$$

(6) The population of a small industrial city has been declining since a major factory closed. The population follows the model $\frac{dP}{dt} = -0.025P$, where P is measured in thousands of people and t is in years. In the year 2020, the population was 48,000.

(a) Find the population function $P(t)$ where $t = 0$ corresponds to the year 2020.

(b) What will be the population in 2030?

(c) In what year will the population drop below 30,000?

(d) What is the half-life of the population (time for the population to decrease by half)?

(a) $P(t) = 48\,e^{-0.025t}$

(b) $P(10) = 48\,e^{-0.25} \approx 37.6$

(c) $48\,e^{-0.025t} = 30$

$t = 40\ln\left(\frac{8}{5}\right)$

(d) $48\,e^{-0.025t} = 24$

$t = 40\ln(2)$.

(x) **Sequences**

(1) Calculate the following limit:

$$\lim_{n\to\infty} (-1)^n \frac{1}{n^2}$$

Does the sequence converge or diverge?

$\lim_{n\to\infty} \frac{1}{n^2} = 0$, Hence $\lim_{n\to\infty} (-1)^n \cdot \frac{1}{n^2} = 0$

Hence $\{(-1)^n \cdot \frac{1}{n^2}\}$ converges to 0

(2) Calculate the following limit:

$$\lim_{n\to\infty} \sin\left(\frac{1}{n}\right)$$

Does the sequence converge or diverge?

$\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$

Hence $\{\sin\left(\frac{1}{n}\right)\}$ converges to 0

(3) Evaluate:

$$\lim_{n\to\infty} \frac{n^2+3n-1}{2n^2-n+5} = \frac{1}{2}$$

Does the sequence converge or diverge?

The sequence converges to $\frac{1}{2}$.

(4) Find the limit:

$$\lim_{n\to\infty} \frac{e^n}{n}$$

Does the sequence converge or diverge?

$$\lim_{n\to\infty} \frac{e^n}{n} = \lim_{x\to\infty} \frac{e^x}{x} \underset{\text{L'H}}{=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

$\frac{\infty}{\infty}$

Hence, the sequence $\left\{\frac{e^n}{n}\right\}$ diverges

(5) Determine whether the following sequence converges, and if so, find its limit:

$$a_n = \frac{n+2}{\ln(n+5)}$$

Does the sequence converge or diverge?

$$\lim_{n\to\infty} \frac{n+2}{\ln(n+5)} = \lim_{x\to\infty} \frac{x+2}{\ln(x+5)} = \lim_{x\to\infty} \frac{1}{\frac{1}{x+5}} = \infty$$

Hence the sequence diverges

(xi) **Series**

Determine whether or not the series below converges or diverges. If it converges, determine to what value.

(1) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+2}\right)$

$$\lim_{k\to\infty} \frac{1}{k} - \frac{1}{k+2} = \lim_{k\to\infty} \frac{\cancel{k}+2-\cancel{k}}{k(k+2)} = 0$$

$$S_1 = 1 - \frac{1}{3}$$

$$S_2 = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4}$$

$$S_3 = 1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \frac{1}{4} + \cancel{\frac{1}{3}} - \frac{1}{5}$$

$$S_4 = 1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \frac{1}{5} + \cancel{\frac{1}{4}} - \frac{1}{6}$$

$$S_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+2}\right) = \lim_{n\to\infty} S_n = 1 + \frac{1}{2}$$

converges to $\frac{3}{2}$

(2) $\sum_{n=2}^{\infty} \ln\left(\frac{n-1}{n+1}\right)$, $\lim_{n\to\infty} \ln\left(\frac{n-1}{n+1}\right) = 0$ ✓

$S_1 = \ln\left(\frac{1}{3}\right) = \ln(1) - \ln(3)$

$S_2 = \ln\left(\frac{1}{3}\right) + \ln\left(\frac{2}{4}\right) = \ln(1) - \ln(3) + \ln(2) - \ln(4)$

$S_3 = \ln\left(\frac{1}{3}\right) + \ln\left(\frac{2}{4}\right) + \ln\left(\frac{3}{5}\right) = \ln(1) - \cancel{\ln(3)} + \ln(2) - \ln(4) + \cancel{\ln(3)} - \ln(5)$

$S_n = \ln(1) + \ln(2) - \ln(n+1) - \ln(n+2)$

$\sum_{n=2}^{\infty} \ln\left(\frac{n-1}{n+1}\right) = \lim_{n\to\infty} \ln(1) + \ln(2) - \ln(n+1) - \ln(n+2)$

$= -\infty$ <u>Diverges</u>.

(3) $\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$

Ignore, it is not included.

(4) $\sum_{n=1}^{\infty} \frac{n^2}{2n^2+5}$ $\lim_{n\to\infty} \frac{n^2}{2n^2+5} = \frac{1}{2} \neq 0$

By the divergency test the series diverges.

(5) $\sum_{n=1}^{\infty} \frac{3n}{n+100}$ $\lim_{n\to\infty} \frac{3n}{n+100} = 3 \neq 0$

By the divergency test the series diverges.

(6) $\sum_{n=1}^{\infty}(-1)^n$

$\lim_{n\to\infty} (-1)^n = \text{DNE} \neq 0$

By the divergency test the series diverges.

(7) $\sum_{n=1}^{\infty}\cos\left(\frac{n\pi}{2n+1}\right)$

Ignore (Not included in the test)

(8) $\sum_{n=1}^{\infty}\frac{2}{2^n}$

Ignore (Not included in the test)

(9) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$

Ignore (Not included in the test)

(10) $\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n}$

$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$

$= 0$

$S_1 = \sqrt{2} - \sqrt{1}$

$S_2 = \cancel{\sqrt{2}} - \sqrt{1} + \sqrt{3} - \cancel{\sqrt{2}} = \sqrt{3} - \sqrt{1}$

$S_3 = \cancel{\sqrt{2}} - \sqrt{1} + \cancel{\sqrt{3}} - \cancel{\sqrt{2}} + \sqrt{4} - \cancel{\sqrt{3}} = \sqrt{4} - \sqrt{1}$

$S_n = \sqrt{n+1} - \sqrt{1}$

$\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n} = \lim_{n\to\infty} \sqrt{n+1} - \sqrt{1} = \infty$

the series diverges

(11) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Ignore (Not included in the test)

(12) $\sum_{n=1}^{\infty} \frac{2n+3}{5n+1}$

Ignore (Not included in the test)

(13) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

Ignore (Not included in the test)