

Integral of $\sec^3 x$

We'll use integration by parts and this ends up being a boomerang integral. We'll also use the fact that

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C$$

$$\int \sec^3 x \, dx = \int \sec x \cdot \sec^2 x \, dx$$

$u = \sec x$	$dv = \sec^2 x \, dx$
$du = \sec x \tan x \, dx$	$v = \tan x$

$$= \sec x \tan x - \int \tan x \cdot \sec x \tan x \, dx$$

$$= \sec x \tan x - \int \sec x \cdot \tan^2 x \, dx$$

$$\text{↖ } \tan^2 x = \sec^2 x - 1$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$$

$$= \sec x \tan x - \int \sec^3 x - \sec x \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \ln |\sec x + \tan x|$$

We have:

$$\begin{aligned} \int \sec^3 x \, dx &= \sec x \tan x - \int \sec^3 x \, dx + \ln |\sec x + \tan x| \\ + \int \sec^3 x \, dx &\quad + \int \sec^3 x \, dx \end{aligned}$$

$$2 \int \sec^3 x \, dx = \sec x \tan x + \ln |\sec x + \tan x| + C$$

$$\Rightarrow \int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C$$