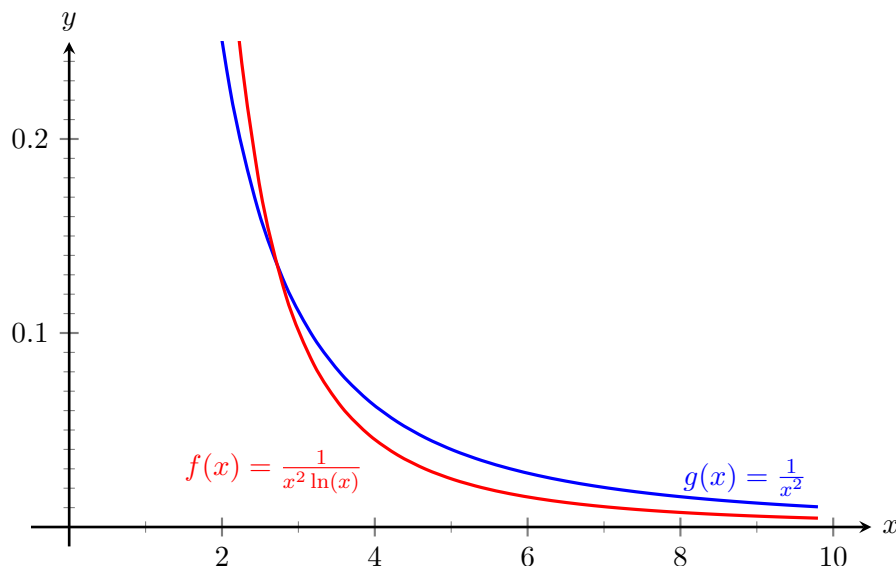


1. The graphs of the functions $f(x) = \frac{1}{x^2 \ln(x)}$ and $g(x) = \frac{1}{x^2}$ are given below. Pay attention to any intersection points and which function has higher values.



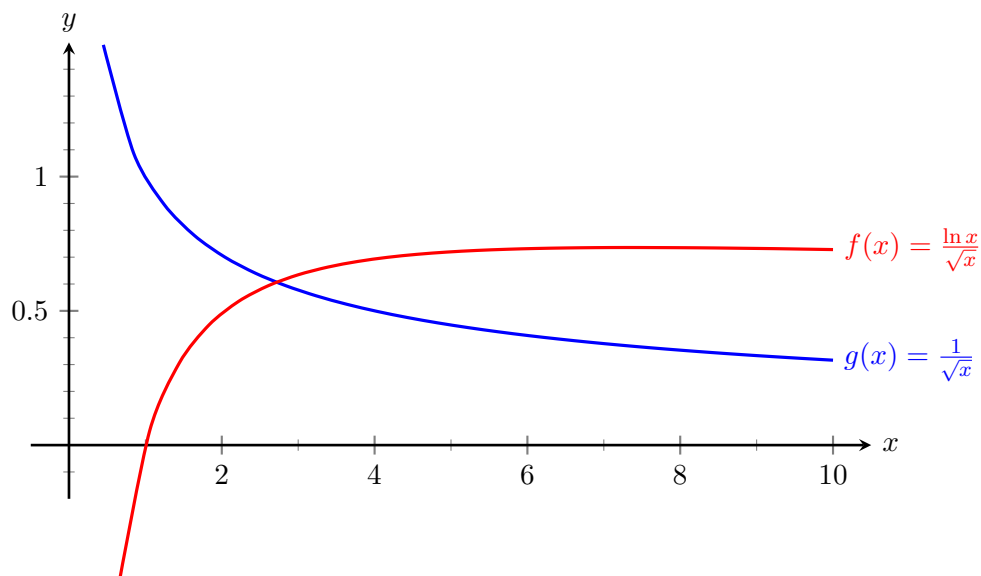
2. (a) Using the graph, do you think $\int_3^\infty \frac{1}{x^2 \ln(x)} dx \leq \int_3^\infty \frac{1}{x^2} dx$ is true? Why or why not?

Solution: Yes. We see from the graphs that on the interval $[3, \infty)$ the graph of $\frac{1}{x^2}$ is above the graph of $\frac{1}{x^2 \ln(x)}$. So the graph of $\frac{1}{x^2 \ln(x)}$ is squeezed between the x -axis and $\frac{1}{x^2}$ which means the area under it and to the right of $x = 3$ must be smaller.

- (b) What does this suggest about the convergence/divergence of the two integrals and why?

Solution: Since $\int_3^\infty \frac{1}{x^2} dx$ converges ($p = 2$ which is greater than 1), $\int_3^\infty \frac{1}{x^2 \ln(x)} dx$ must also converge. This is because of the inequality in (a): if the larger integral is finite, the smaller integral must also be finite.

3. The graphs of the functions $f(x) = \frac{\ln(x)}{\sqrt{x}}$ and $g(x) = \frac{1}{\sqrt{x}}$ are given below.



Do the functions $f(x)$ and $g(x)$ have any horizontal asymptotes? How do you know?

4. (a) Using the graph, do you think $\int_3^\infty \frac{1}{\sqrt{x}} dx \leq \int_3^\infty \frac{\ln(x)}{\sqrt{x}} dx$ is true? Why or why not?

Solution: Yes, because on $[3, \infty)$ the graph of $\frac{\ln(x)}{\sqrt{x}}$ lies above the graph of $\frac{1}{\sqrt{x}}$.

- (b) What does this suggest about the convergence/divergence of the two integrals and why?

Solution: $\int_3^\infty \frac{1}{\sqrt{x}} dx$ diverges (since $p = 1/2$ which is less than 1), so the integral $\int_3^\infty \frac{\ln(x)}{\sqrt{x}} dx$ must also diverge because it is larger.

The punchline:**The Comparison Theorem for Improper Integrals**

If f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$, then

(a) If $\int_a^\infty f(x) dx$ is convergent, then $\int_a^\infty g(x) dx$ is convergent.

(b) If $\int_a^\infty g(x) dx$ is divergent, then $\int_a^\infty f(x) dx$ is divergent.

If we know $\int_a^\infty f(x) dx$ diverges, what can we conclude about $\int_a^\infty g(x) dx$? **Solution:** Nothing.

If we know $\int_a^\infty g(x) dx$ converges, what can we conclude about $\int_a^\infty f(x) dx$? **Solution:** Nothing.

Putting it into practice:

5. Consider the integral $\int_2^\infty \frac{\cos^2(x)}{x^2} dx$.

(a) A good comparison function is:

Solution: $\frac{1}{x^2}$

(b) What is the relationship between your comparison function and $\frac{\cos^2(x)}{x^2}$?

Solution: $\frac{\cos^2(x)}{x^2} \leq \frac{1}{x^2}$ since $\cos^2(x) \leq 1$

(c) What conclusion can you make using the Comparison Theorem?

Solution: Since $\int_2^\infty \frac{1}{x^2} dx$ is convergent ($p = 2 > 1$), we can say that $\int_2^\infty \frac{\cos^2(x)}{x^2} dx$ also converges.

6. Consider the integral $\int_1^{\infty} \frac{1}{\sqrt{x^2 + 1}} dx$.

(a) A good comparison function is:

Solution: $1/x$

(b) What is the relationship between your comparison function and $\frac{1}{\sqrt{x^2 + 1}}$?

Solution: $\frac{1}{\sqrt{x^2 + 1}} \leq \frac{1}{x}$ since $x \leq \sqrt{x^2 + 1}$

(c) Will this inequality help you determine if $\int_1^{\infty} \frac{1}{\sqrt{x^2 + 1}} dx$ is convergent or divergent?

Solution: It won't help. Knowing the integral is smaller than a divergent integral tells us nothing.