

Method 1:

Let W_t^x be a standard Brownian motion starting at $x \in (a, b)$, and killed upon exiting the interval (a, b) . The transition density of this process, given in spectral form, is found in (Borodin and Salminen, *Handbook of Brownian Motion*, p. 126):

$$p(t; x, y) = \frac{2}{b-a} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi(x-a)}{b-a}\right) \sin\left(\frac{n\pi(y-a)}{b-a}\right) \exp\left(-\frac{n^2\pi^2}{2(b-a)^2}t\right)$$

This is the conditional density that $W_t^x = y$, assuming the process has not exited the interval up to time t . The survival probability up to time t , i.e., the probability that $W_s \in (a, b)$ for all $s \leq t$, is

$$\mathbb{P}_x(T > t) = \int_a^b p(t; x, y) dy.$$

The time derivative of the survival probability is the survival density, so the time derivative of $(1 - \text{the survival probability})$ is the exit time density. Let $f_{(a,b)}(t, x)$ be the density of the first exit time from the interval (a, b) for a Brownian motion started at x . Then

$$\begin{aligned} f_{(a,b)}(t, x) &= \frac{d}{dt}(1 - \mathbb{P}_x(T \leq t)) \\ &= -\frac{d}{dt}\mathbb{P}_x(T > t) \\ &= -\frac{d}{dt} \int_a^b p(t; x, y) dy \\ &= -\frac{d}{dt} \int_a^b \left[\frac{2}{b-a} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi(x-a)}{b-a}\right) \sin\left(\frac{n\pi(y-a)}{b-a}\right) \exp\left(-\frac{n^2\pi^2}{2(b-a)^2}t\right) \right] dy \\ &= -\sum_{n=1}^{\infty} \frac{2}{b-a} \sin\left(\frac{n\pi(x-a)}{b-a}\right) \exp\left(-\frac{n^2\pi^2}{2(b-a)^2}t\right) \cdot \frac{d}{dt} \left[\int_a^b \sin\left(\frac{n\pi(y-a)}{b-a}\right) dy \right] \\ &= \sum_{n=1}^{\infty} \frac{2}{b-a} \sin\left(\frac{n\pi(x-a)}{b-a}\right) \left(\frac{n^2\pi^2}{2(b-a)^2}\right) \exp\left(-\frac{n^2\pi^2}{2(b-a)^2}t\right) \cdot \int_a^b \sin\left(\frac{n\pi(y-a)}{b-a}\right) dy \\ &= \sum_{n=1}^{\infty} \frac{2}{b-a} \sin\left(\frac{n\pi(x-a)}{b-a}\right) \left(\frac{n^2\pi^2}{2(b-a)^2}\right) \exp\left(-\frac{n^2\pi^2}{2(b-a)^2}t\right) \cdot \left(\frac{2(b-a)}{n\pi} \cdot (-1)^{n+1}\right) \\ &= \sum_{n=1}^{\infty} \frac{4n\pi}{(b-a)^2} \cdot \sin\left(\frac{n\pi(x-a)}{b-a}\right) \cdot (-1)^{n+1} \cdot \exp\left(-\frac{n^2\pi^2}{2(b-a)^2}t\right). \end{aligned}$$

Going further, the exit time distribution function $F(t)$ is given by

$$\begin{aligned} F(t) &= \int_0^t f_{(a,b)}(s, x) ds \\ &= \int_0^t \sum_{n=1}^{\infty} \frac{4n\pi}{(b-a)^2} \cdot \sin\left(\frac{n\pi(x-a)}{b-a}\right) \cdot (-1)^{n+1} \cdot \exp\left(-\frac{n^2\pi^2}{2(b-a)^2}s\right) ds \\ &= \sum_{n=1}^{\infty} \frac{4n\pi}{(b-a)^2} \cdot \sin\left(\frac{n\pi(x-a)}{b-a}\right) \cdot (-1)^{n+1} \int_0^t \exp\left(-\frac{n^2\pi^2}{2(b-a)^2}s\right) ds \\ &= \sum_{n=1}^{\infty} \frac{4n\pi}{(b-a)^2} \cdot \sin\left(\frac{n\pi(x-a)}{b-a}\right) \cdot (-1)^{n+1} \cdot \left(\frac{1 - \exp\left(-\frac{n^2\pi^2}{2(b-a)^2}t\right)}{\frac{n^2\pi^2}{2(b-a)^2}}\right) \\ &= \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \cdot \sin\left(\frac{n\pi(x-a)}{b-a}\right) \cdot \left(1 - \exp\left(-\frac{n^2\pi^2}{2(b-a)^2}t\right)\right). \end{aligned}$$

When we run simulation with this F by taking say 50, 100, 150 terms of the sum, the empirical distribution does not match. This could be due to numerical/convergence issues?

Method 2:

Here we make two changes: We use the transition density wrt speed measure and we do not analytically do the integration for F . Instead we let python do the numerical integration.

Let W_t^x be a standard Brownian motion starting at $x \in (a, b)$, and killed upon exiting the interval (a, b) . The transition density of this process (wrt speed measure) is found in (Borodin and Salminen, *Handbook of Brownian Motion*, p. 126):

$$p(t; x, y) = \frac{1}{2\sqrt{2\pi t}} \sum_{n=-\infty}^{\infty} \left[\exp\left(-\frac{(x - y + 2n(b - a))^2}{2t}\right) - \exp\left(-\frac{(x + y - 2a + 2n(b - a))^2}{2t}\right) \right].$$

This is the conditional density that $W_t^x = y$, assuming the process has not exited the interval up to time t . The survival probability up to time t , i.e., the probability that $W_s \in (a, b)$ for all $s \leq t$, is

$$\begin{aligned} \mathbb{P}_x(T > t) &= \int_a^b p(t; x, y) m(dy) \\ &= 2 \int_a^b p(t; x, y) dy. \end{aligned}$$

The exit probability up to time t is $1 -$ the survival probability up to time t so

$$\begin{aligned} F(t) &= 1 - \mathbb{P}_x(T > t) \\ &= 1 - 2 \int_a^b p(t; x, y) dy. \end{aligned}$$

Then the numerical calculation (python) for F is as follows.

```
1 def p_txy(t, x, y, a, b, n_terms):
2     L = b - a
3     s = 0
4     for n in range(1, n_terms + 1):
5         coeff = (2 / L)
6         sine_x = np.sin(n * np.pi * (x - a) / L)
7         sine_y = np.sin(n * np.pi * (y - a) / L)
8         decay = np.exp(-(n**2 * np.pi**2 * t) / (2 * L**2))
9         s += coeff * sine_x * sine_y * decay
10    return s
11
12 S_vals = []
13 for t in t_vals:
14     p_vals = p_txy(t, x0, y_vals, a, b, n_terms)
15     S = np.trapezoid(p_vals, y_vals)
16     S_vals.append(S)
17
18 S_vals = np.array(S_vals)
19 F_vals = 1 - S_vals
```

Listing 1: Numerical integration for computing $F(t, x)$

