

For problems 1–4, we are working with $f(x) = e^x$. We'll use different approximations of f near 0 to approximate the numbers $e^{0.1}$ and e .

1. **Linear Approximation:** We want to find a linear function $L(x) = C_0 + C_1 x$ whose value and slope at $x = 0$ matches the value and slope of f at $x = 0$.

(a) Fill in the table below:

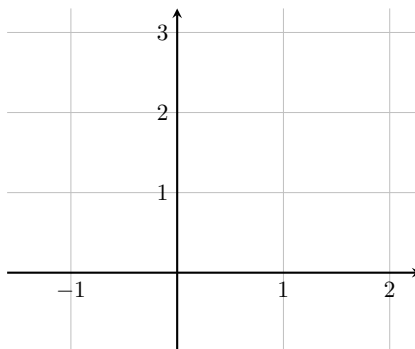
$f(x) = e^x$	$f(0) =$	$L(x) = C_0 + C_1 x$	$L(0) =$
$f'(x) =$	$f'(0) =$	$L'(x) =$	$L'(0) =$

(b) Find the coefficients of L .

- If $f(x)$ and $L(x)$ have the same value at 0, then $f(0) = L(0)$ so $C_0 =$ _____
- If $f(x)$ and $L(x)$ have the same slope at 0, then $f'(0) = L'(0)$ so $C_1 =$ _____

Thus $L(x) =$ _____

(c) Graph e^x and $L(x)$.



(d) Use $L(x)$ to estimate $e^{0.1}$ and e .

(e) Plug $e^{0.1}$ and e into a calculator and compare these values to the estimates you got in part (d). How accurate are your estimates?

Note: The general formula for a linear approximation centered at $x = 0$ is $L(x) = f(0) + f'(0)x$. This is the equation of the tangent line to f at 0. It is also called the **1st degree Taylor Polynomial** for $f(x)$ centered at $x = 0$. Confirm that the answer you got for $L(x)$ in part (b) matches this formula.

2. **Quadratic Approximation:** We want to find a quadratic function $Q(x) = C_0 + C_1x + C_2x^2$ whose value, slope and concavity at $x = 0$ matches that of f at $x = 0$.

(a) Fill in the table below.

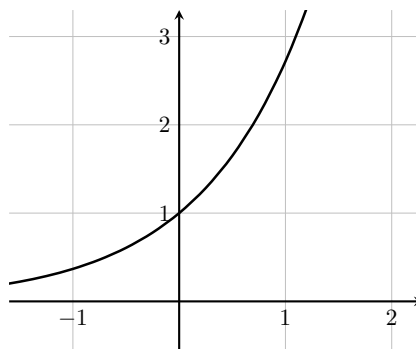
$f(x) = e^x$	$f(0) = 1$	$Q(x) = C_0 + C_1x + C_2x^2$	$Q(0) =$
$f'(x) =$	$f'(0) =$	$Q'(x) =$	$Q'(0) =$
$f''(x) =$	$f''(0) =$	$Q''(x) =$	$Q''(0) =$

(b) Find the coefficients of Q .

- If $f(x)$ and $Q(x)$ have the same value at 0, then $f(0) = Q(0)$ so $C_0 =$ _____
- If $f(x)$ and $Q(x)$ have the same slope at 0, then $f'(0) = Q'(0)$ so $C_1 =$ _____
- If $f(x)$ and $Q(x)$ have the same concavity at 0, then $f''(0) = Q''(0)$ so $2C_2 =$ _____, hence $C_2 =$ _____

Thus $Q(x) =$ _____

(c) A graph of e^x is shown below. Graph $Q(x)$ on the same axes.



(d) Use $Q(x)$ to estimate $e^{0.1}$ and e .

(e) Compare the estimates you got in part (d) to the values you got from your calculator. Explain why the errors are reduced.

Note: The general formula for a quadratic approximation of f centered at $x = 0$ (AKA the **2nd degree Taylor polynomial** of f centered at $x = 0$) is:

$$Q(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2.$$

Check that your answer for $Q(x)$ in part (b) matches this formula.

3. **Cubic Approximation:** We want to find a cubic function $T(x) = C_0 + C_1x + C_2x^2 + C_3x^3$ whose value and whose first, second and third derivatives at $x = 0$ match those of f at $x = 0$.

(a) Fill in the table below:

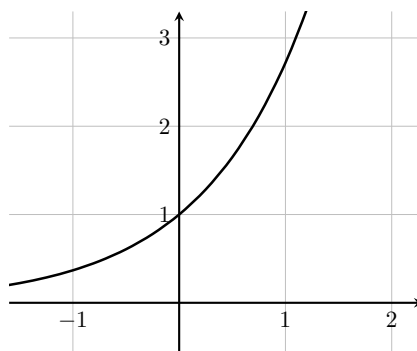
$f(x) = e^x$	$f(0) = 1$	$T(x) = C_0 + C_1x + C_2x^2 + C_3x^3$	$T(0) = C_0$
$f'(x) =$	$f'(0) =$	$T'(x) =$	$T'(0) =$
$f''(x) =$	$f''(0) =$	$T''(x) =$	$T''(0) =$
$f'''(x) =$	$f'''(0) =$	$T'''(x) =$	$T'''(0) =$

(b) Find the coefficients of T .

- If $f(0) = T(0)$, then $C_0 =$ _____
- If $f'(0) = T'(0)$, then $C_1 =$ _____
- If $f''(0) = T''(0)$, then _____
- If $f'''(0) = T'''(0)$, then _____

Thus $T(x) =$ _____

(c) A graph of e^x is shown below. Graph $T(x)$ on the same coordinate plane.



(d) Use $T(x)$ to estimate e^{-1} and e . How does the accuracy compare to that of the previous two problems?

Note: The general formula for a cubic approximation of f centered at $x = 0$ (AKA the **3rd degree Taylor polynomial** of f centered at $x = 0$) is:

$$T(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3.$$

Check that your answer for $T(x)$ in part (b) matches this formula.

4. (a) Fill in the table below, for $f(x) = e^x$. Note that $f^{(n)}(x)$ means the n th derivative of $f(x)$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$	$\frac{f^{(n)}(0)}{n!} x^n$
0				
1				
2				
3				
4				
5				
6				

- (b) Add the terms in the last column to get the 6th degree Taylor Polynomial for $f(x) = e^x$ centered at $x = 0$.

- (c) Use the 6th degree Taylor polynomial to estimate e .

5. Write down the general formula for the n th degree Taylor polynomial for a function $f(x)$, centered at $x = 0$.

6. Give an explanation of where the factorials come from.