LECTURES ON STOCHASTIC HOMOGENIZATION

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Abstract. The content of this lecture is the theory of stochastic homogenization for divergence-form operators. The first part deals with linear operators, while the second part generalizes to quasilinear equations. These notes were written for a mini-course consisting of three lectures at Tohoku University, Sendai, Japan.

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Contents

1. Introduction 2
2. Stochastic homogenization for linear divergence-form equations 2
2.1. Intuition and equations 2
2.2. Connection to SDEs 4
2.3. Examples for media with random impurities 4
2.4. A formal computation 6
2.5. Assumtions, ergodicity and wellposedness 6
2.6. The candidate limit 9
2.7. Stochastic motivation for the corrector 10
2.8. The random homogenization corrector 11
2.9. The homogenized coefficient 17
2.10. Proof of the homogenization result 18
2.11. Strong convergence of the two-scale expansion 21
3. Stochastic homogenization for nonlinear divergence-form equations 23
3.1. Assumptions, ergodicity and wellposedness 24
3.2. Homogenization and Gamma convergence 25
3.3. The random homogenization corrector 26
3.4. The homogenized coefficient 27
3.5. Proof of the homogenization result 27
References 28

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1. Introduction

Homogenization describes the limit procedure from microscropic boundary-value problems posed on periodic or random structures to a macroscopic equation.
The theory was developed from engineering purposes to understand complicated microstructures such as composite materials or porous media in the 1970s, cf. the classical book [11]. First, the theory of periodic homogenization was studied ([11]), where the material is assumed to have certain periodic structures. An example of a periodic material could be a material consisting of periodic tiles like a checkerboard (eg. simple conductor-insulator models in electromagnetics). However, closer to reality might be to assume that the material is almost periodic or has random impurities, that could be deposited like a random tiling or random points (eg. Poisson clouds, random clusters). If the scale of impurities is much smaller compared to the size of the region, where the system is posed, then an asymptotic analysis is needed. Homogenization thus describes the passage from the microscale description to the macroscale description of the system. In the microscale description we encounter a family of equations with oscillating coefficients indexed by the ratio of microscopic and macroscopic scales $\varepsilon \in (0,1)$. When $\varepsilon \to 0$, we pass to the macroscopic description of the system and observe homogenized coefficients. Interestingly, the coefficients of that limiting system are in general not just given by the averaged coefficients and some nontrivial effects happen.
In this lecture, we assume that the reader is familiar with elliptic and parabolic PDEs (cf. eg. [3, 12]). Knowledge in stochastic analysis (martingales, stochastic differential equations, Itô's formula, [13]) and the theory of periodic homogenization (for a recap, see for example [4, section 2]) is helpful, but not necessarily needed. Indeed, periodic homogenization can be seen as a special case of stochastic homogenization and is thus also covered, although certain proof steps are actually less involved in the periodic case. Section 2 is based on the lecture notes by Benjamin Fehrman [4]. The methods originated from [7] and were unified in [10]. Section 3 is based on content from the book [9]. We want to point out that there is a vast literature on stochastic homogenization (see for example [1] and the references therein) and we can only cover a small part of it in this short lecture.

2. Stochastic homogenization for linear divergence-form equations

Our prime example in this section are equations in divergence form, which can model the conduction of heat or electricity. Non-divergence form equations in homogenization are significantly more involved (and in fact in the stochastic case, not much is known). For simplicity of the presentation we only consider elliptic equations with zero boundary conditions. The below arguments can also be used to prove the corresponding results for the parabolic equations.

2.1. Intuition and equations.

We consider the heat flow on a bounded, open domain $U \subseteq \mathbb{R}^d$ through a random impure medium, that is, for some diffusion matrix $a(x,\omega)$ defined on $\mathbb{R}^d\times\Omega$, where $(\Omega,\mathscr{F},\mathbb{P})$ denotes the probability space, the density $u(t,x)$ of heat/charge evolves according to the parabolic equation

(2.1)
$$\partial_t u = \nabla\cdot(a(x,\omega)\nabla u) + f \text{ on } U\times(0,\infty), \quad u = g \text{ on } \partial U\times(0,\infty), \quad u = u_0 \text{ on } U\times\{0\}$$

and in its steady state (formally $\partial_t u = 0$) according to the elliptic equation

$$-\nabla\cdot(a(x,\omega)\nabla u) = f \text{ on } U, \quad u = g \text{ on } \partial U. \tag{2.2}$$

Here f denotes a source term. The matrix a describes the directions in which heat gets dissolved and is taken to be random. The case of $a(x,\omega)\equiv \mathrm{Id}$ corresponds to the case of perfectly homogeneous material. We then study the microscropic equations

(2.3)
$$\partial_t u^\varepsilon = \nabla\cdot(a(x/\varepsilon,\omega)\nabla u^\varepsilon) + f \text{ on } U\times(0,\infty), \quad u^\varepsilon = g \text{ on } \partial U\times(0,\infty), \quad u^\varepsilon = u_0 \text{ on } U\times\{0\}$$

and

$$-\nabla\cdot(a(x/\varepsilon,\omega)\nabla u^\varepsilon) = f \text{ on } U, \quad u^\varepsilon = g \text{ on } \partial U, \tag{2.4}$$

where $\varepsilon\in(0,1)$ denotes the microscale of impurities. In stochastic homogenization, we ask about the convergence of (u^ε) to a solution $\bar u$ of an effective equation with deterministic, i.e. none-random, coefficient $\bar a$. The limiting equation describes the dynamics from the point of view of zooming out (macrostructure perspective). The solution $\bar u$ shall solve

$$\partial_t \bar u = \nabla\cdot(\bar a\nabla u^\varepsilon) + f \text{ on } U\times(0,\infty), \quad \bar u = g \text{ on } \partial U\times(0,\infty), \quad u = u_0 \text{ on } U\times\{0\}, \tag{2.5}$$

respectively,

$$-\nabla\cdot(\bar a\nabla\bar u) = f \text{ on } U, \quad \bar u = g \text{ on } \partial U. \tag{2.6}$$

Notice that we proposed that the divergence-form structure of the equation is being preserved in the limit. We aim to determine the effective coefficient $\bar a$, as well as study the convergence (weak, strong, pointwise convergence?)

$$u^\varepsilon \to \bar u \text{ as } \varepsilon\to 0.$$

In quantitative homogenization one moreover studies the convergence speed of (u^ε) to $\bar u$, which is particularily relevant for numerics. In the case of stochastic coefficients, this is a rather new field and we refer to the book [1]. In this lecture, we will however only focus on qualitative convergence statements.

A special case of the random setting is the periodic setting (and also variants like the almost periodic case etc.). Indeed, let $(\Omega,\mathscr{F},\mathbb{P}) = (\mathbb{T}^d,\mathscr{B}(\mathbb{T}^d),\lambda)$ for the normalized Lebesgue measure λ on the torus $\mathbb{T}^d=\mathbb{R}^d/\mathbb{Z}^d$ and $a(x,\omega) := \tilde a(\tau_x(\omega))$ for the shift $\tau_x:\Omega\to\Omega$, $\tau_x(\omega)=\omega+x$

mod 1, $x \in \mathbb{R}^d$ and $\tilde{a} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ being 1-periodic. Then $\tilde{a}(\cdot/\varepsilon)$ is ε-periodic and ε is the periodicity scale of the microstructure.

2.2. **Connection to SDEs.**
In the case of symmetric a with $\sigma = \sqrt{2a}$ the PDE (2.1) corresponds to the solution X of the SDE

$$dX_t^x = (\nabla \cdot a)(X_t,\omega)dt + \sigma(X_t,\omega)dB_t, \quad X_0^x = x \in \mathbb{R}^d \tag{2.7}$$

for a d-dimensional Brownian motion B on a different filtered probability space $(\tilde{\Omega}, (\tilde{\mathscr{F}}_t), \tilde{\mathbb{P}})$. Indeed, by the Feynman-Kac formula, we have that if u solves (2.1), then

$$u(t,x,\omega) = \tilde{\mathbb{E}}[u_0(X_t^x) \mid \tau > t] + \tilde{\mathbb{E}}[g(X_\tau^x) \mid \tau \leq t] + \tilde{\mathbb{E}}\bigg[\int_0^{\tau\wedge t} f(X_s^x)ds\bigg] \tag{2.8}$$

and vice versa, where $\tilde{\mathbb{E}} = \int d\tilde{\mathbb{P}}$ denotes integration with respect to the probability measure $\tilde{\mathbb{P}}$ and τ denotes the first exit time of the process X from U. This reflects the fact that the infinitesimal generator, that determines the law of the solution X to the SDE (2.7), is given by the operator

$$\mathscr{L}(\omega)f = \nabla \cdot (a(x,\omega)\nabla f), \quad f \in C_c^\infty.$$

Due to the scaling properties of the Brownian motion, one sees that the generator of the rescaled process $(\varepsilon X_{t/\varepsilon^2}^{x/\varepsilon})_{t\geqslant 0}$ is the operator $\mathscr{L}^\varepsilon(\omega)$ with coefficient $a(x/\varepsilon,\omega)$ and thus

$$u^\varepsilon(t,x) = \tilde{\mathbb{E}}[u_0(\varepsilon X_{t/\varepsilon^2}^{x/\varepsilon}) \mid \tau^\varepsilon > t] + \tilde{\mathbb{E}}[g(\varepsilon X_{\tau/\varepsilon^2}^{x/\varepsilon}) \mid \tau^\varepsilon \leq t] + \tilde{\mathbb{E}}\bigg[\int_0^{\tau^\varepsilon\wedge t} f(\varepsilon X_{s/\varepsilon^2}^{x/\varepsilon})ds\bigg].$$

Fixing $\omega \in \Omega$, the central limit theorem for the convergence of the processes $((\varepsilon X_{t/\varepsilon^2}^{x/\varepsilon})_{t\geqslant 0})_\varepsilon$ in law to a process $\bar{X}$ on the probability space $(\tilde{\Omega}, (\tilde{\mathscr{F}}_t), \tilde{\mathbb{P}})$, where $\bar{X}$ solves the SDE

$$d\bar{X}_t^x = (\nabla \cdot \bar{a})dt + \bar{\sigma}dB_t, \quad \bar{X}_0^x = x$$

with $\bar{\sigma} = \sqrt{2\bar{a}}$, thus implies the convergence of $(u^\varepsilon(t,x,\omega))_\varepsilon$ to $\bar{u}(t,x,\omega)$, where $\bar{u}$ solves the equation for $\bar{\mathscr{L}}$ with coefficient $\bar{a}$. We refer to the book [6, section 9.6] for results in this direction.

2.3. **Examples for media with random impurities.**
In this section, we will construct random environments, that model a material with randomly deposited impurities. The environments $a(x,\omega)$ will be bounded in (x,ω), spatially stationary, that is $(a(x,\cdot))_x$ and $(a(x+y,\cdot))_x$ have the same law for any y, as well as sufficiently mixing, see in Assumption 2.3 below for the precise definitions.

Example 2.1. *The first material we construct is the random checkerboard model. The space is paved in unit-size cubes and each cube is colored either black or white independently at* *random. That is, let $(b(z))_{z\in\mathbb{Z}^d}$ be independent random variables defined on $(\Omega,\mathscr{F},\mathbb{P})$ with $\mathbb{P}(b(z)=0)=\mathbb{P}(b(z)=1)=1/2$. Here, z stands for the middle point of the cube. Let a_0,a_1 be (non-random) uniformly elliptic matrices. Then the random environment is defined by*

$$a(x,\omega)=a_{b(z)(\omega)},\quad x\in z+[-1/2,1/2)^d, \tag{2.9}$$

so in each cube with middle point z it either takes the value a_0 or a_1 depending on the realization $b(z)$.

Example 2.2. *The second material we construct has impurities modelled by a Poisson point process. Let Ω denote the space of all locally finite point measures on $\mathbb{R}^d$, that is, the collection of all measures of the form*

$$\omega=\sum_{j\in\mathscr{J}}\delta_{x_j},$$

where $\mathscr{J}\subseteq\mathbb{N}$ and $x_j\in\mathbb{R}^d$ for $j\in\mathscr{J}$ and δ_{x_j} denotes the Dirac measure in x_j, and such that $\omega(B)=\#\{j\in\mathscr{J}\mid x_j\in B\}$ is finite for any bounded Borel set B. Let $X^B:\Omega\to\mathbb{R},\omega\mapsto\omega(B)$ and

$$\mathscr{F}:=\sigma(\{X^B\mid B\text{ bounded Borel}\}),$$

that is, the filtration generated by all X^B for bounded Borel $B\subset\mathbb{R}^d$, which makes them measurable maps with respect to $\mathscr{F}$. Let $\lambda\in(0,\infty)$. We now uniquely define a probability measure $\mathbb{P}_\lambda$ on $(\Omega,\mathscr{F})$ by the following three properties:

1. *For every bounded Borel set B, $\mathbb{E}_\lambda[X^B]=\lambda|B|$, where $|B|$ denotes the Lebesgue measure of B.*
2. *For every collection $B_1,\dots B_N$ of bounded and pairwise disjoint Borel sets and any $N\in\mathbb{N}$, $(X^{B_1},\dots,X^{B_k})$ are independent random variables.*
3. *For every $y\in\mathbb{R}^d$, $f:\mathbb{R}^N\to\mathbb{R}$ bounded and measurable and $B_1,..,B_N$ bounded, Borel, $N\in\mathbb{N}$, $\mathbb{E}_\lambda[f(X^{B_1},...,X^{B_N})]=\mathbb{E}_\lambda[f(X^{B_1+y},...,X^{B_N+y})]$.*

The measure $\mathbb{P}_\lambda$ is called the Poisson point measure with intensity λ and $(X^B)_{B\text{ bounded, Borel}}$ is called the Poisson point process with intensity λ. Property (1) implies that on average, there are λ points in a set of Lebesgue measure 1. Property (2) implies a strong mixing property of the environment, meaning that the behaviour of the process in disjoint sets is independent. Property (3) implies a spatial stationarity property, which means that the environment is statistically homogeneous. One can show that for a bounded Borel B, $\mathbb{P}_\lambda(X^B=n)=\frac{(\lambda|B|)^n}{n!}e^{-\lambda|B|}$, $n\in\mathbb{N}$, such that indeed X^B is Poisson distributed with parameter $\lambda|B|$.
Let again a_0,a_1 be uniformly elliptic. For a realization $\omega=\sum_{j\in\mathscr{J}}\delta_{x_j}$, we define the random coefficient field as follows:

$$a(x,\omega)=a_0\mathbf{1}_{\cup_{j\in\mathscr{J}}B_{1/2}(x_j)}(x)+a_1\mathbf{1}_{\mathbb{R}^d\setminus\cup_{j\in\mathscr{J}}B_\delta(x_j)}(x). \tag{2.10}$$

That is, in the $1/2$*-neighbourhood of each point* x_j *in the realization of the point process, we see the diffusion coefficient* a_0*, while outside of that neighbourhoods we see* a_1*. If* $a_0 \simeq 0$ *and* $a_1 \simeq \mathrm{Id}$*, this can model a medium with low conductivity in the Poisson points and high conductivity everywhere else. In place of balls* $B_{1/2}(x_j)$ *also other shapes are conceivable. Moreover environments, where the radius of these balls is itself random are conceivable, but more complicated.*

2.4. **A formal computation.**

We start with a motivating, but at this stage formal, computation for the elliptic equation. Let $d = 1$. From the equation (2.2), we obtain after integration and assuming that a is bounded and positive and $g = 0$, that the solution is given by

$$\nabla u^\varepsilon(x,\omega) = a^{-1}(x/\varepsilon,\omega)\Big(c - \int_0^x f(y)dy\Big).$$

By the ergodic theorem, since a^{-1} is stationary and ergodic, we obtain that for almost all ω,

$$a^{-1}(x/\varepsilon,\omega) \rightharpoonup \langle a^{-1}\rangle = \mathbb{E}[a^{-1}(0,\cdot)]$$

weakly in $L^2_{loc}(\mathbb{R})$ (and in general not strong). Hence setting $\bar{a} = \langle a^{-1}\rangle^{-1}$, we obtain that $\nabla u^\varepsilon \rightharpoonup \nabla\bar{u}$ with the convergence being weak in $L^2_{loc}(U)$. Notice that in general, $\langle a^{-1}\rangle^{-1} \neq \langle a\rangle$, unless a is constant. This means that the divergence form structure of the equation is conserved after homogenization, but the coefficient does not equal the averaged coefficient in general. The same argument applies in the periodic case, where the ergodic theorem corresponds to the fact that for a bounded, 1-periodic a, $a^{-1}(\cdot/\varepsilon) \rightharpoonup \langle a^{-1}\rangle = \int_0^1 a^{-1}dx$ weakly in $L^2_{loc}(\mathbb{R})$. We will see that in $d \geq 2$ there exists no such simple formula for the homogenized coefficient.

2.5. **Assumtions, ergodicity and wellposedness.**

In what follows we give the needed definitions and tools to rigorously prove the stochastic homogenization result for the elliptic equation.

Assumption 2.3. *(Random environment) The coefficient field* $a : \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$ *is defined on a probability space* $(\Omega, \mathscr{F}, \mathbb{P})$ *and is assumed to be measurable. By switching to the image measure of* a *under* $\mathbb{P}$*, we assume without loss of generality that* $(\Omega, \mathscr{F}, \mathbb{P})$ *is such that* Ω *is the space of all environments* $\omega : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ *and* a *is the canonical process, that is* $a(x,\omega) = \omega(x)$*. Then we define the group of shift transformations* $\tau_x : \Omega \to \Omega$ *by* $\tau_x\omega = \omega(\cdot - x)$ *for* $x \in \mathbb{R}^d$ *(having* $\tau_{x+y} = \tau_x \circ \tau_y$*). The environment and the shift are assumed to satisfy:*

(1) *uniform ellipticity: there exist* $\Lambda, \lambda \in (0,\infty)$ *such that almost surely, for every* $x, \xi \in \mathbb{R}^d$*,*

$$|a(x,\omega)\xi| \leq \Lambda|\xi|, \quad a(x,\omega)\xi\cdot\xi \geq \lambda|\xi|^2$$

(2) *stationarity/measure-preservation of the shift: for all* $A \in \mathscr{F}$, $\mathbb{P}(A) = \mathbb{P}(\tau_x A)$ *and* $a(x,\omega) = \tilde{a}(\tau_x \omega)$, *where* $\tilde{a}(\omega) = a(0,\omega)$

(3) *ergodicity: for all* $A \in \mathscr{F}$ *with* $\tau_x A = A$ *for all* $x \in \mathbb{R}^d$ *(sets, that are invariant under shifts) it follows that* $\mathbb{P}(A) = 0$ *or* $\mathbb{P}(A) = 1$

(4) *stochastic continuity: for every* $\delta \in (0,1)$,

$$\lim_{x\to 0} \mathbb{P}(\{\omega \mid |a(x,\omega) - a(0,\omega)| > \delta\}) = 0.$$

Remark 2.4. *The environments* (2.9), (2.10) *and the periodic environment, that is* $a(x)$ *being* 1*-periodic, continuous and uniformly elliptic, satisfy the above requirements.*
Ergodicity is for example implied by the stronger condition of finite range dependence: for any two subsets $A, B \subseteq \mathbb{R}^d$ *with strictly positive distance from each other,* $\sigma(a(x,\cdot) \mid x \in A)$ *and* $\sigma(a(x,\cdot) \mid x \in B)$ *are independent.*
The stochastic continuity is a technical condition, which ensures that the process can be taken to be separable, jointly measurable in (x,ω) *and* $\mathscr{F}$ *is countably generated.*
Having the measure-preserving shifts (τ_x), *we can define the corresponding group of unitary operators* (T_x) *by* $T_x : L^2(\Omega,\mathscr{F},\mathbb{P}) \to L^2(\Omega,\mathscr{F},\mathbb{P})$, $T_x f(\omega) = f(\tau_{-x}\omega)$. *One can show that this group is strongly continuous.*

In the following we state a version of Birkhoff's ergodic theorem. It tells us, that under an ergodic transformation, spatial averages can be replaced by averages on the probability space.

Theorem 2.5. *(Ergodic theorem) Let* $(\Omega,\mathscr{F},\mathbb{P})$ *be a probability space and let* (τ_x) *be an ergodic (i.e. property (3)), measure-preserving (i.e. property (2)) group of transformations on* Ω. *Then for every* $f \in L^1(\Omega,\mathscr{F},\mathbb{P})$, *there exists a set* $\tilde{\Omega}$ *of measure* 1, *such that for* $\omega \in \tilde{\Omega}$ *and any bounded, open subset* $U \subset \mathbb{R}^d$ *with* $0 \in U$ *and* $U_R := \{Rx \mid x \in U\}$,

$$\mathbb{E}[f] = \lim_{R\to\infty} \frac{1}{|U_R|} \int_{U_R} f(\tau_x\omega)dx.$$

Corollary 2.6. *Let* $(\Omega,\mathscr{F},\mathbb{P})$ *be a probability space and let* (τ_x) *be an ergodic (i.e. property (3)), measure-preserving (i.e. property (2)) group of transformations on* Ω. *Let* $p \in (1,\infty)$. *Then for any* $f \in L^p(\Omega)$, *for almost all* ω,

$$f(\tau_{x/\varepsilon}\omega) \rightharpoonup \mathbb{E}[f]$$

weakly in $L^p_{loc}(\mathbb{R}^d)$ *as* $\varepsilon \to 0$.

Proof. By density of simple functions (linear combinations of indicator functions) built from open, bounded sets that contain the origin in $L^{p'}_{loc}$ for p' conjugate to p, it suffices to prove that for any bounded and open $U \subset \mathbb{R}^d$ with $0 \in U$,

$$\int_U f(\tau_{x/\varepsilon}\omega)dx \to |U|\mathbb{E}[f].$$

Taking $R = \varepsilon^{-1}$, substituting $x/\varepsilon \to x$ and noticing that $|U_R| = R^d|U|$, the claim follows from the ergodic theorem. □

In particular, in the setting of the corollary for $p = 2$, almost sure local L^2-boundedness follows, that is for almost all ω,

$$\sup_{\varepsilon\in(0,1)} \|f(\tau_{x/\varepsilon}\omega)\|_{L^2(B_R)} < \infty.$$

Let us get back to the homogenization problem and first answer the wellposedness question. The proof of the proposition is classical and follows from the Lax-Milgram lemma, that we state here for convenience. Below $H_0^\beta(U)$ denotes the Sobolev space of regularity $\beta \geq 0$ with zero boundary conditions.

Lemma 2.7. *(Lax-Milgram) Let H be a Hilbert space and $B : H \times H \to \mathbb{K}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ be a bilinear form, which is continuous, that is there exists a constant $\Lambda \in (0,\infty)$, such that $|B(u, v)| \leq \Lambda\|u\|\|v\|$ for all $u, v \in H$, and which is coercive, that is there exists a constant $\lambda \in (0,\infty)$ such that $B(u, u) \geq \lambda\|u\|^2$ for all $u \in H$. Then for any $f \in H^*$ (the dual of H), there exists a unique $u \in H$ such that*

$$B(u, v) = f(v) \text{ for all } v \in H.$$

Moreover the following bound holds:

$$\|u\| \leq \lambda^{-1}\|f\|_{H^*}.$$

Proposition 2.8. *Let $U \subset \mathbb{R}^d$ bounded, open and $a : U \times \Omega \to \mathbb{R}^{d\times d}$ be uniformly elliptic in the sense of property (1) and let $f \in H^{-1}(U)$. Then almost surely there exists a unique weak solution u to* (2.2)*, that is, for almost all ω, $u(\omega) \in H_0^1(U)$ and for any $\psi \in H_0^1(U)$,*

$$\int a(x, \omega)\nabla u(x, \omega) \cdot \nabla\psi(x)dx = \int f(x)\psi(x)dx.$$

Furthermore, there exists a constant $C > 0$, such that almost surely the solution satisfies the bound

$$\|u\|_{H_0^1(U)} \leq C\|f\|_{H^{-1}(U)}.$$

Proof. Fixing ω out of the set of full measure, where a satisfies property (1), we can apply the Lax-Milgram lemma to the following continuous bilinear form $B : H_0^1(U) \times H_0^1(U) \to \mathbb{R}$ with

$$B(u, v) = \int a(x, \omega)\nabla u(x) \cdot \nabla v(x)dx.$$

Continuity follows from Hölder's inequality and the upper bound on a. Due to the lower bound on a and Poincaré inequality, we have that

$$|B(u, u)| \geq \lambda\|\nabla u\|^2_{L^2(U)} \geq c\lambda\|u\|^2_{H_0^1(U)},$$

with $\lambda c > 0$, such that B is coercive. Thus for any $f \in H^{-1}(U)$, there exists a unique solution $u(\omega)$ to $B(u(\omega), \psi) = \langle f, \psi \rangle$ for all $\psi \in H_0^1(U)$ and u satisfies the bound with $C = 1/c\lambda$. $\square$

In particular, for the solutions u^ε of

$$\nabla \cdot (a(x/\varepsilon, \omega)\nabla u^\varepsilon) = f \text{ on } U, \quad u^\varepsilon = 0 \text{ at } \partial U,$$

we obtain that, almost surely

$$\sup_{\varepsilon \in (0,1)} \|u^\varepsilon\|_{H_0^1(U)} \leq C\|f\|_{H^{-1}(U)}. \tag{2.11}$$

Since $H_0^1(U)$ is reflexive and thus bounded sets are compact in the weak topology, this yields that almost surely (u^ε) converges along subsequences in the weak topology. Next, we identify possible limit points.

2.6. **The candidate limit.**

To derive a candidate for a limit of (u^ε), we employ multiscale analysis, that is a formal argument making the ansatz

$$u^\varepsilon(x, \omega) = u_0(x, x/\varepsilon, \omega) + \varepsilon u_1(x, x/\varepsilon, \omega) + \varepsilon^2 u_2(x, x/\varepsilon, \omega) + \dots,$$

for $u_0, u_1, u_2, \dots$ to be determined. The same ansatz is done in the periodic case, however here it is random. We let $y = x/\varepsilon$ be the fast variable. Then we have that

$$\nabla u_i(x, y, \omega) = \nabla_x u_i(x, y, \omega) + \varepsilon^{-1}\nabla_y u_i(x, y, \omega), \quad i = 0, 1, 2, \dots$$

and plugging into the equation for u^ε tested against a test function $\psi \in C_c^\infty(U)$, we find

$$\begin{aligned}\int_U a(x/\varepsilon, \omega)\nabla u^\varepsilon(x, \omega) \cdot \nabla\psi(x)dx &= \int_U a(x/\varepsilon, \omega)(\nabla_x u_0 + \varepsilon^{-1}\nabla_y u_0 + \nabla_y u_1 + O(\varepsilon)) \cdot \nabla\psi(x)dx \\ &= \int_U f(x)\psi(x)dx.\end{aligned}$$

As $\varepsilon \to 0$, we see that u_2 and higher order terms do not effect the weak formulation. Therefore they can be set to zero and we are left to determine u_0, u_1. Notice that this exploits that the equation is in divergence-form, otherwise there would be higher order terms. We then obtain that

$$-[\nabla_x + \varepsilon^{-1}\nabla_y] \cdot (a(y, \omega)[\nabla_x u_0 + \nabla_y u_1 + \varepsilon^{-1}\nabla_y u_0]) = f.$$

To determine u_0, u_1, we equate the terms of the same order in ε. On the level of ε^{-2}, we obtain

$$-\nabla_y \cdot (a(y, \omega)\nabla_y u_0(x, y, \omega)) = 0.$$

Due to the energy bound $\|\nabla_y u_0\|_{L^2}^2 \leq \lambda^{-1} \int a\nabla_y u_0 \nabla_y u_0 dy = 0$, it follows that $u_0(x, y, \omega)$ needs to be constant in y and thus we can drop the dependence on the fast variable. Hence,

$u_0(x,\omega)$ is our candidate limit. At the order ε^{-1}, we obtain the equation

$$-\nabla_y \cdot (a(y,\omega)[\nabla_x u_0 + \nabla_y u_1]) = 0.$$

Making a separation of variables ansatz of the form $u_1(x,y,\omega) = \phi_i(y,\omega)\partial_i u_0(x,\omega)$, employing Einstein summation here and below, we obtain

$$-\nabla_y \cdot [a(y,\omega)(e_i + \nabla_y \phi_i(y,\omega))]\partial_i u_0 = 0.$$

Letting ϕ solve the so-called cell/corrector equation

$$-\nabla_y \cdot [a(y,\omega)(e_i + \nabla_y \phi_i(y,\omega))] = 0, \quad y \in \mathbb{R}^d, \tag{2.12}$$

the latter equation is also fulfilled. The equation of order 1 is given by, after plugging $y = x/\varepsilon$,

$$-\nabla_x \cdot (a(x/\varepsilon,\omega)(e_i + \nabla_y \phi_i(x/\varepsilon,\omega))\partial_i u_0) = f.$$

Assuming that $\nabla\phi$ is stationary and using Corollary 2.6, we see that almost surely

$$a(x/\varepsilon,\omega)(e_i + \nabla\phi_i(x/\varepsilon,\omega)) \rightharpoonup \mathbb{E}[a(0,\cdot)(e_i + \nabla\phi_i(0,\cdot))] =: \bar{a},$$

with $\bar{a}$ being deterministic. Thus, u_0 solves

$$-\nabla \cdot (\bar{a}\nabla u_0) = f \tag{2.13}$$

and is in fact deterministic. We learnt that the homogenized coefficient is given by

$$\bar{a} = \mathbb{E}[a(0,\cdot)(e_i + \nabla\phi_i(0,\cdot))],$$

where ϕ solves the corrector/cell equation (2.12). Moreover, we saw that the ansatz

$$u^\varepsilon(x,\omega) = u_0(x) + \varepsilon\phi_i(x/\varepsilon,\omega)\partial_i u_0(x) \tag{2.14}$$

gives the correct prediction of the limit $\bar{u} = u_0$ when $\varepsilon \to 0$.
Although the above arguments are purely formal, for the proof of the homogenization result, the ansatz (2.14) turns out to be useful. Note that we will need to solve the cell equation (2.12) to make use of the ansatz. In the periodic case, the cell equation (2.12) for $y \in \mathbb{T}^d$ can be solved using the classical theory based on Lax-Milgram. The solution will be periodic and is unique in the class of solutions with zero mean. However, in the random case, solving the cell equation (2.12) turns out to be more involved. In particular, although the coefficient a is stationary, the solution ϕ is in general not stationary.

2.7. **Stochastic motivation for the corrector.**

On the level of the diffusion process X solving (2.7), the corrector ϕ has a different motivation. Indeed, we see from an application of Ito's formula that (after first mollifying ϕ to make it sufficiently regular and passing to the limit)

$$\begin{aligned}\phi(X_t,\omega)-\phi(X_0,\omega) &= \int_0^t \mathscr{L}(\omega)\phi(X_s,\omega)ds + \int_0^t \nabla\phi(X_s,\omega)\sigma(X_s,\omega)dB_s \\ &= X_0 - X_t + \int_0^t (\nabla\phi(X_s,\omega)+\mathrm{Id})\sigma(X_s,\omega)dB_s,\end{aligned}$$

where the last equality holds if we choose ϕ, such that

$$\mathscr{L}(\omega)\phi(x,\omega) = -\nabla\cdot a(x,\omega), \quad \mathscr{L}(\omega)=\nabla\cdot(a(x,\omega)\nabla\cdot).$$

This is the rewritten cell Equation (2.12). In the CLT scaling, we thus obtain

$$\varepsilon(X_{t/\varepsilon^2}-X_0) = \varepsilon(\phi(X_0,\omega)-\phi(X_{t/\varepsilon^2},\omega)) + \varepsilon M_{t/\varepsilon^2}, \tag{2.15}$$

where M is the martingale

$$M_t = \int_0^t (\nabla\phi(X_s,\omega)+\mathrm{Id})\sigma(X_s,\omega)dB_s.$$

That means, in (2.15) we have written X as a martingale plus a boundary term. The boundary term vanishes under the CLT scaling. Thus, the previous problem of proving a central limit theorem for X is reduced to a martingale central limit theorem for M. Since martingales essentially equal time-changed Brownian motions in law (by the Dubins-Schwarz theorem), the central limit theorem for martingales is classical and the limit is given by a Brownian motion with constant diffusivity $\bar{a}$, where $\bar{a}$ is the limit of the quadratic variations $\langle \varepsilon M_{\cdot/\varepsilon^2}\rangle_1$ (convergence by the ergodic theorem). This strategy goes back to the work by Kipnis-Varadhan [5], cf. also the book [6] for all the powerful applications. We have proven that these methods can even be applied in a setting of very singular coefficients, cf. [8].

2.8. **The random homogenization corrector.**

To solve the corrector equation (2.12) in the case of random coefficient a, we need to use the probabilistic structure, that is the ergodicity of the coefficient field. Simply fixing ω results in a loss of compactness, which in the periodic case is obtained by the compact state space. Compactness in the stochastic case is obtained via ergodicity.

Let $(\Omega,\mathscr{F},\mathbb{P})$ be as in Assumption 2.3 with Ω being the space of environments $\omega:\mathbb{R}^d\to\mathbb{R}^{d\times d}$ and the canonical process $a(x,\omega)=\omega(x), x\in\mathbb{R}^d$ with $a(x,\omega)=T_x a(0,\omega)=T_x\tilde{a}(\omega)$. Let (τ_x) be measure-preserving, that is property (2) shall be satisfied.

To lift the equation to the probability space, we have to define a different notion of derivatives and Sobolev spaces. Therefore, let the horizontal derivative be defined as follows. Let $f\in L^2(\Omega)$ and let for $i=1,\dots,d$ (e_i denoting the euclidean unit vector),

$$D_i f(\omega) := \lim_{h\to 0}\frac{f(\tau_{he_i}\omega)-f(\omega)}{h},$$

whenever the limit in $L^2(\Omega)$ exists. The domain $\mathscr{D}(D_i)$ of D_i consists of those $f \in L^2(\Omega)$, for which the latter limit exists. One can show that the operators D_i are densely defined and closed. In fact, $D = (D_1, \dots, D_d)$ is the infinitesimal generator of the shift operator group (T_x), i.e. $Df = \partial_x T_x f_{|x=0}$.
For $f \in L^2(\Omega)$, we may associate the stationary process $\tilde{f}(x, \omega) = T_x f(\omega)$. Ergodicity in this context means that the only translation invariant $f \in L^2(\Omega)$, i.e. with $T_x f = f$, $\forall x$, are the constant functions.
Let the "Sobolev space on the probability space" be defined by

$$\mathscr{H}^1(\Omega) = \cap_{i=1}^d \mathscr{D}(D_i)$$

equipped with the inner product

$$\langle f, g \rangle_{\mathscr{H}^1} = \mathbb{E}[fg] + \sum_{i=1}^d \mathbb{E}[D_i f D_i g].$$

$\mathscr{H}^1(\Omega)$ is a Hilbert space, since the D_i are closed operators. We define $\mathscr{H}^{-1}$ as the dual of $\mathscr{H}^1$, that is the space of linear and bounded maps $u : \mathscr{H}^1 \to \mathbb{R}$. The gradients (D_i) are curl-free and have zero mean (due to the shifts being measure preserving), that is

$$D_i D_j \psi = D_j D_i \psi \quad \text{and} \quad \mathbb{E}[D_i \psi] = 0, \quad \psi \in \mathscr{H}^1(\Omega).$$

The adjoint of D_i is $-D_i$, that is $\mathbb{E}[D_i f g] = -\mathbb{E}[f D_i g]$ for $f, g \in \mathscr{H}^1$, since the adjoint of T_x is T_{-x}.
Notice that there is no Poincaré inequality on the probability space, such that the $\mathscr{H}^1$-norm is not simply equivalent to the semi-norm defined by $\mathbb{E}[Df Df]$, as it is the case for $H_0^1(U)$ on a bounded, open domain U. In particular, the form $B(u, v) := \mathbb{E}[aDu \cdot Dv]$ defined on $\mathscr{H}^1 \times \mathscr{H}^1$ is in general not coercive. But it becomes coercive as soon as we add $+\beta\mathbb{E}[uv]$ for $\beta > 0$. This means that instead of $\mathscr{L}$ we consider the resolvent operator $\mathscr{L} - \beta \, \mathrm{Id}$. Thus, instead of solving the Poisson equation/cell equation (2.12) in $\mathscr{H}^1$ (which can only be solved under stronger assumptions like a spectral gap etc., cf.[6]), we solve instead the resolvent equation $(\mathscr{L} - \beta)\Phi^\beta = -D \cdot a$ in $\mathscr{H}^1$. Indeed, the coercivity condition can also be interpreted as a spectral gap and by introducing $\beta > 0$, we enforce a spectral gap. In particular, the resolvent equation is always solvable. We will then prove that $(D\Phi^\beta)_\beta$ converges to a stationary field. To that aim, define the space of potential vector fields

$$L^2_{pot}(\Omega) = \overline{\{f \in L^2(\Omega, \mathbb{R}^d) \mid f = D\psi, \psi \in \mathscr{H}^1(\Omega)\}}^{L^2}$$

As noted above, potential vector fields are curl-free fields an have mean zero. For $f \in L^2(\Omega)$, we define $Df \in \mathscr{H}^{-1}$ as the distributional derivative satisfying $\mathbb{E}[Dfg] = -\mathbb{E}[fDg]$ for any

$g \in \mathscr{H}^1$. Then we define the solenoidal/divergence-free vector fields

$$L^2_{sol}(\Omega) = \{f \in L^2(\Omega, \mathbb{R}^d) \mid \sum_{i=1}^d D_i f = 0\}.$$

One can show that $L^2_{pot}(\Omega)^\perp = L^2_{sol}(\Omega)$ (indeed this follows from $\mathbb{E}[fg] = \mathbb{E}[f]\mathbb{E}[g]$ for $f \in L^2_{pot}(\Omega), g \in L^2_{sol}(\Omega)$, which follows from the div-curl Lemma 2.18 below and the ergodic theorem, Corollary 2.6) and thus a Helmholtz-type decomposition holds $L^2(\Omega, \mathbb{R}^d) = L^2_{pot}(\Omega) \oplus L^2_{sol}(\Omega)$.

Proposition 2.9. *Let $(\Omega, \mathscr{F}, \mathbb{P})$, (τ_x) and a satisfy properties (1), (2) in Assumption 2.3. Let $\beta > 0$. Then for any $g \in \mathscr{H}^{-1}$, there exists a unique solution $\Phi^{\beta,g} \in \mathscr{H}^1$ to the resolvent equation $-D \cdot (aD\Phi^{\beta,g}) + \beta\Phi^{\beta,g} = g$, that is*

$$\mathbb{E}[aD\Phi^{\beta,g} D\psi] + \beta\mathbb{E}[\Phi^{\beta,g}\psi] = \mathbb{E}[g\psi], \quad \forall \psi \in \mathscr{H}^1(\Omega). \tag{2.16}$$

If $g = D \cdot \tilde{g}$ for $\tilde{g} \in L^2(\Omega)$, the solutions satisfy

$$\sup_{\beta \in (0,1)} \|D\Phi^{\beta,g}\|_{L^2} \leq \lambda^{-1}\|\tilde{g}\|_{L^2}, \qquad \sup_{\beta \in (0,1)} \beta\|\Phi^{\beta,g}\|^2_{L^2} \leq \lambda^{-1}\|\tilde{g}\|^2_{L^2}. \tag{2.17}$$

Proof. The wellposedness follows from the Lax-Milgram lemma together with coercitivity and continuity of the bilinear form $B(u,v) = \mathbb{E}[aDu \cdot Dv] + \beta\mathbb{E}[uv]$ defined on $\mathscr{H}^1 \times \mathscr{H}^1$. The bounds follow from the equation tested against $\Phi^{\beta,g}$, property (1) on a and using that $g = D \cdot \tilde{g}$:

$$\begin{aligned}\beta\mathbb{E}[\Phi^{\beta,g}\Phi^{\beta,g}] + \lambda\mathbb{E}[D\Phi^{\beta,g} \cdot D\Phi^{\beta,g}] &\leq \beta\mathbb{E}[\Phi^{\beta,g}\Phi^{\beta,g}] + \mathbb{E}[aD\Phi^{\beta,g} \cdot D\Phi^{\beta,g}] \\ &= -\mathbb{E}[\tilde{g}D\Phi^{\beta,g}] \\ &\leq \mathbb{E}[|\tilde{g}|^2]^{1/2}\mathbb{E}[|D\Phi^{\beta,g}|^2]^{1/2}\end{aligned}$$

and thus

$$\lambda\|D\Phi^{\beta,g}\|_{L^2(\Omega)} \leq \|\tilde{g}\|_{L^2(\Omega)},$$

such that

$$\beta\|\Phi^{\beta,g}\|^2_{L^2} \leq \mathbb{E}[|\tilde{g}|^2]^{1/2}\mathbb{E}[|D\Phi^{\beta,g}|^2]^{1/2} \leq \lambda^{-1}\|\tilde{g}\|_{L^2(\Omega)}.$$

□

We can apply the proposition to the right-hand-side $g_i = -D \cdot ae_i \in \mathscr{H}^{-1}$, which yields solutions that we denote by Φ^β_i for $i = 1, \ldots, d$. By taking the limit $\beta \to 0$ for $(D\Phi^\beta)_\beta$, we recover solutions to the equation for the gradient of the corrector. That gradient is by construction stationary, but it turns out that the corrector itself won't be stationary in general. The next proposition proves the wellposedness of the equation for the gradient.

Proposition 2.10. *Let $(\Omega, \mathscr{F}, \mathbb{P})$, (τ_x) and a satisfy properties (1), (2) in Assumption 2.3. Then for each $i = 1, \ldots, d$, there exists a unique $\Psi_i \in L^2_{pot}(\Omega)$ satisfying*

$$\mathbb{E}[a(\Psi_i + e_i) \cdot \varphi] = 0, \quad \forall \varphi \in L^2_{pot}(\Omega). \tag{2.18}$$

Proof. The claim follows from the Lax-Milgram lemma applied to the bilinear form $B(u, v) = \mathbb{E}[au \cdot v]$ defined on $L^2_{pot}(\Omega) \times L^2_{pot}(\Omega)$. □

Corollary 2.11. *Let $(\Omega, \mathscr{F}, \mathbb{P})$, (τ_x) and a satisfy properties (1), (2) in Assumption 2.3. Let for each $i = 1, \ldots, d$, Φ_i^β solve the resolvent equation with right-hand-side $g = -D \cdot ae_i$. Then $(D\Phi_i^\beta)$ converges weakly in $L^2(\Omega)$ to the solution Ψ_i of (2.18), $i = 1, \ldots, d$.*

Proof. Due to the bounds (2.17), we obtain weak convergence of $(D\Phi_i^\beta)$ and $(\beta^{1/2}\Phi_i^\beta)$ in $L^2(\Omega)$ along subsequences. The limit of $(D\Phi_i^\beta)$ along such a subsequence is in $L^2_{pot}(\Omega)$, since the space is closed. Passing to the limit in the equation for Φ_i^β tested against $\psi \in \mathscr{H}^1(\Omega)$, we obtain that the limit of any subsequence of $(D\Phi_i^\beta)$ solves (2.18). Since the equation (2.18) is uniquely solvable in $L^2_{pot}(\Omega)$, we obtain convergence along the whole sequence and that the limit equals Ψ_i. □

Via the transformation group, we can define a process on the physical space that corresponds to Ψ on the probability space. We derive the following properties on the process.

Proposition 2.12. *Let $(\Omega, \mathscr{F}, \mathbb{P})$, (τ_x) and a satisfy properties (1), (2) in Assumption 2.3. Let $\Psi_i = (\Psi_{ik})_{k=1,\ldots,d}$ be given as in Proposition 2.10 and define the process $\psi(x, \omega) = \Psi(\tau_x\omega)$, $x \in \mathbb{R}^d$. Then it follows that almost surely for all i, $\psi_i \in L^2_{loc}(\mathbb{R}^d; \mathbb{R}^d)$ and for the distributional derivatives holds for almost all (x, ω),*

$$\partial_j \psi_{ik}(x, \omega) = \partial_k \psi_{ij}(x, \omega).$$

In particular, it follows that almost surely $\psi_i \in L^2_{pot}(\mathbb{R}^d) = \{\psi : \mathbb{R}^d \to \mathbb{R}^d \mid \psi = \nabla f, f \in H^1_{loc}(\mathbb{R}^d)\}$.

Proof. Due to Fubini's theorem and stationary, we have that

$$\mathbb{E}\left[\int_{B_R} |\psi_i(x, \cdot)|^2 dx\right] = \int_{B_R} \mathbb{E}[|\Psi_i|^2] dx = |B_R| \mathbb{E}[|\Psi_i|^2].$$

This implies that there exists a one-set $\tilde{\Omega}_R$ with $\|\psi_i(\cdot, \omega)\|_{L^2(B_R, \mathbb{R}^d)} < \infty$ for all $\omega \in \Omega_R$. Due to continuity of $R \mapsto \|\psi_i(\cdot, \omega)\|_{L^2(B_R, \mathbb{R}^d)}$, we can find a one-set $\tilde{\Omega}$, such that $\|\psi_i(\cdot, \omega)\|_{L^2(B_R, \mathbb{R}^d)} < \infty$ for all $\omega \in \tilde{\Omega}$ and all $R > 0$. Hence it follows that almost surely $\psi_i \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}^d)$. The equality of the derivatives follows from the following two observations. Let $\psi^\beta(x, \omega) = D\Phi^\beta(\tau_x\omega) = \Psi^\beta(\tau_x\omega)$. First, we observe that Ψ^β satisfies the curl-free condition

$$D_j \Psi_{ik}^\beta = D_j D_k \Phi_i^\beta = D_k D_j \Phi_i^\beta = D_k \Psi_{ij}^\beta.$$

Secondly, we use this to see that

$$
\begin{aligned}
\mathbb{E}\Big[1_A \int \partial_j \psi_{ik}(x,\cdot)\phi(x)dx\Big] &\leftarrow \mathbb{E}\Big[1_A \int \partial_j \psi^\beta_{ik}(x,\cdot)\phi(x)dx\Big] \\
&= -\mathbb{E}\Big[1_A \int \psi^\beta_{ik}(x,\cdot)\partial_j\phi(x)dx\Big] \\
&= -\mathbb{E}\Big[D_k\Phi^\beta_i \int 1_A(\tau_{-x})\partial_j\phi(x)dx\Big] \\
&= -\mathbb{E}\Big[D_k\Phi^\beta_i D_j \int 1_A(\tau_{-x}\cdot)\phi(x)dx\Big] \\
&= \mathbb{E}\Big[D_j D_k\Phi^\beta_i \int 1_A(\tau_{-x}\cdot)\phi(x)dx\Big] \\
&= \mathbb{E}\Big[D_k D_j\Phi^\beta_i \int 1_A(\tau_{-x}\cdot)\phi(x)dx\Big] \\
&= \mathbb{E}\Big[1_A \int \partial_k \psi^\beta_{ij}(x,\cdot)\phi(x)dx\Big] \\
&\to \mathbb{E}\Big[1_A \int \partial_k \psi_{ij}(x,\cdot)\phi(x)dx\Big]
\end{aligned}
$$

for any $A \in \mathscr{F}$, $\phi \in C_c^\infty(\mathbb{R}^d)$, due to weak convergence of $(D\Phi^\beta = \Psi^\beta)$ to Ψ in $L^2(\Omega)$ by Corollary 2.11. □

Combining the latter propositions yields the existence of the correctors.

Theorem 2.13. *Let $(\Omega, \mathscr{F}, \mathbb{P})$, (τ_x) and a satisfy properties (1), (2) in Assumption 2.3. Let ψ be given as in Proposition 2.12. Then for every $i = 1, \dots, d$, there exists a unique $\phi_i \in H^1_{loc}(\mathbb{R}^d)$ satisfying the properties: almost surely*

$$
|B_1|^{-1} \int_{B_1} \phi_i(x,\omega) = 0, \quad \nabla\phi_i = \psi_i \tag{2.19}
$$

and almost surely for all $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$
\int a(x,\omega)(\nabla\phi_i(x,\omega) + e_i)\cdot\nabla\varphi(x)dx = 0. \tag{2.20}
$$

Proof. Let by the previous proposition, $\tilde\phi_i \in H^1_{loc}(\mathbb{R}^d)$ be such that $\nabla\tilde\phi_i = \psi_i$ and let $\phi_i := \tilde\phi_i - |B_1|^{-1}\int_{B_1}\tilde\phi_i(x,\cdot)dx$. Then $\nabla\phi_i = \nabla\tilde\phi_i = \psi_i$ and ϕ_i has mean zero on the ball B_1. Moreover, the property (2.19) determines ϕ_i uniquely, since the only $H^1_{loc}(\mathbb{R}^d)$ function ϕ that satisfies $\int_{B_1}\phi dx = 0$ and $\nabla\phi = 0$ is the zero function. It remains to prove that ϕ_i satisfies the equation. We have that for any $A \in \mathscr{F}$, $\varphi \in C_c^\infty$, using that almost surely $\nabla\phi_i = \psi_i$ and

$\psi(x,\omega) = \Psi(\tau_x\omega)$,

$$\begin{aligned}\mathbb{E}\Big[1_A \int a(x,\cdot)(\nabla\phi_i(x,\cdot)+e_i)\cdot\nabla\varphi(x)dx\Big] &= \mathbb{E}\Big[1_A \int a(\tau_x\cdot)(\Psi_i(\tau_x\cdot)+e_i)\cdot\nabla\varphi(x)dx\Big]\\ &= \mathbb{E}\Big[a(0,\cdot)(\Psi(\cdot)+e_i)\cdot\int 1_A(\tau_{-x}\cdot)\nabla\varphi(x)dx\Big]\\ &= \mathbb{E}\Big[a(0,\cdot)(\Psi_i+e_i)\cdot D\int 1_A(\tau_{-x}\cdot)\varphi(x)dx\Big]=0,\end{aligned}$$

where in the last line we used that $D\cdot(a(0,\cdot)(\Psi_i+e_i))=0$, since Ψ solves (2.18). Since A was arbitrary, we obtain that there exists a one-set Ω_φ (may depend on φ), such that

$$\int a(x,\omega)(\nabla\phi_i(x,\omega)+e_i)\cdot\nabla\varphi(x)dx=0$$

for all $\omega\in\Omega_\varphi$. Due to separability of C_c^∞, we can find a one-set $\tilde\Omega$, such that

$$\int a(x,\omega)(\nabla\phi_i(x,\omega)+e_i)\cdot\nabla\varphi(x)dx=0$$

for all $\varphi\in C_c^\infty$ and all $\omega\in\tilde\Omega$. □

We note that ϕ_i is in general not stationary. Indeed stationarity would imply that, since $\int_{B_1}\phi_i dx=0$, also $\int_{B_1(y)}\phi_i dx=0$ for all $y\in\mathbb{R}^d$, which is not true in general.
Recall the multiscale ansatz

$$u^\varepsilon(x/\varepsilon,\omega)=u_0(x)+\varepsilon\phi_i(x/\varepsilon,\omega)\partial_i u_0(x),$$

where u_0 solves the effective equation

$$-\nabla\cdot(\bar a\nabla u_0)=f.$$

In the case where ϕ_i is 1-periodic in the spatial variable, it follows that $(\phi_i(x/\varepsilon))$ converges weakly in L^2_{loc} to $\int_0^1\phi_i(x)dx$ and thus the remainder $(\varepsilon\phi_i(x/\varepsilon)\partial_i u_0)$ vanishes strongly in L^2_{loc}. Hence, one concludes that u_0 is indeed the limit of (u^ε). In the random case, one cannot directly apply the ergodic theorem to $\phi_i(x/\varepsilon,\omega)$ due to the lack of stationarity. However, one can show that ϕ_i is sublinear, which implies that the remainder in the multiscale expansion vanishes. This is the content of the next proposition.

Proposition 2.14. *Let Assumption 2.3 be satisfied. Let ϕ_i be defined as in Theorem 2.13 with $\nabla\phi_i=\psi$ and $\psi(x,\omega)=\Psi(\tau_x\omega)$ being stationary. Let B be an open ball. Then almost surely*

$$\lim_{\varepsilon\to0}\int_B|\varepsilon\phi_i(x/\varepsilon,\cdot)|^2dx=0.$$

We will instead prove the following equivalent statement (the proof of the equivalence can be checked in [4, Proposition 3.7]), which will be sufficient for our purpose, since we can add constants to the correctors solving Equation (2.12).

Proposition 2.15. *Let Assumption 2.3 be satisfied. Let ϕ_i be defined as in Theorem 2.13 with $\nabla\phi_i = \psi_i$ and $\psi(x,\omega) = \Psi(\tau_x\omega)$ being stationary. Let B be an open ball. Then almost surely*

$$\limsup_{\varepsilon\to 0}\left(\varepsilon\phi_i(x/\varepsilon,\cdot) - |B|^{-1}\int_B \varepsilon\phi_i(x/\varepsilon,\cdot)dx\right) = 0$$

with convergence in $L^2(B)$.

Proof. By the ergodic theorem, Corollary 2.6, we have that

$$\Psi_i(\tau_{x/\varepsilon}\cdot) \rightharpoonup \mathbb{E}[\Psi_i] = 0 \tag{2.21}$$

weakly in $L^2_{loc}(\mathbb{R}^d;\mathbb{R}^d)$, where the expectation vanishes, since Ψ is a potential field. Using that

$$\nabla[\varepsilon\phi_i(x/\varepsilon,\cdot)] = (\nabla\phi_i)(x/\varepsilon,\cdot) = \Psi(\tau_{x/\varepsilon},\cdot)$$

and the Poincaré inequality on B we obtain that in particular almost surely

$$\sup_{\varepsilon\in(0,1)}\left\|\left(\varepsilon\phi_i(x/\varepsilon,\cdot) - |B|^{-1}\int_B \varepsilon\phi_i(x/\varepsilon,\cdot)dx\right)\right\|_{H^1(B)} < \infty.$$

Thus we obtain weak convergence in $H^1(B)$ along subsequences. Since the embedding $H^1(B) \hookrightarrow L^2(B)$ is compact, the convergence of subsequences is also strong in $L^2(B)$. Each limit φ needs to equal 0, since $\nabla\varphi = 0$ due to (2.21) and $\int_B \varphi(x)dx = 0$. Thus the convergence is along the whole sequence and the limit is $\varphi = 0$. ☐

2.9. **The homogenized coefficient.**

Let the effective coefficient $\bar{a}$ be defined as follows

$$\bar{a}e_i = \mathbb{E}[a(0,\cdot)(\Psi_i + e_i)] \tag{2.22}$$

for Ψ solving (2.18). We will show that $\bar{a}$ is also uniformly elliptic. This ensures the well-posedness of the effective equation.

Proposition 2.16. *Let a satisfy Assumption 2.3 and let Ψ be given as in Proposition 2.10. Then $\bar{a}$ defined in* (2.22) *satisfies*

$$|\bar{a}\xi| \le \Lambda\left(\sum_{i=1}^d \mathbb{E}[|\Psi_i + e_i|^2]\right)^{1/2}|\xi|, \quad \bar{a}\xi\cdot\xi \ge \lambda|\xi|^2, \quad \forall\xi\in\mathbb{R}^d.$$

Proof. The upper bound follows immediately from the upper bound on a and Hölder's inequality. The lower bound follows using the lower bound on a, the equation for Ψ in (2.18), Jensen's inequality and $\mathbb{E}[\Psi] = 0$ such that

$$\begin{aligned}\bar{a}\xi\cdot\xi = \mathbb{E}[a(0,\cdot)(\Psi_\xi + \xi)\cdot\xi] &= \mathbb{E}[a(0,\cdot)(\Psi_\xi + \xi)\cdot(\Psi_\xi + \xi)]\\ &\ge \lambda\mathbb{E}[|\Psi_\xi + \xi|^2] \ge \lambda|\mathbb{E}[\Psi_\xi + \xi]|^2 = \lambda|\xi|^2,\end{aligned}$$

where $\Psi_\xi = \xi_i\Psi_i$. ☐

Proposition 2.17. *Let Assumption 2.3 be satisfied and let* Ψ *be given as in Proposition 2.10. Let* Ψ^t *solve* (2.18) *for* a *replaced by the transpose* a^t*. Let* $\bar{a}$ *be given by* (2.22) *and let* $\tilde{a} = \mathbb{E}[a^t(0,\cdot)(\Psi_i^t + e_i)]$*. Then* $\tilde{a} = (\bar{a})^t$ *and* $\bar{a}$ *is symmetric if* a *is symmetric.*

Proof. For each $i, j = 1, \dots, d$, we have that, using the equations for Ψ, Ψ^t,

$$\begin{aligned}\bar{a}_{ij} = \mathbb{E}[a(0,\cdot)(\Psi_i + e_i)] \cdot e_j &= \mathbb{E}[a(0,\cdot)(\Psi_i + e_i) \cdot (\Psi_j^t + e_j)] \\ &= \mathbb{E}[(\Psi_i + e_i) \cdot a^t(0,\cdot)(\Psi_j^t + e_j)] = \tilde{a}_{j,i}\end{aligned}$$

Finally, if $a = a^t$ is symmetric, then $\Psi = \Psi^t$ by uniqueness and thus $\tilde{a} = \bar{a}$, such that $\bar{a} = (\bar{a})^t$. □

2.10. Proof of the homogenization result.

To prove the convergence $u^\varepsilon \rightharpoonup \bar{u}$ in $H_0^1(U)$, where $\bar{u}$ solves the equation with effective coefficient $\bar{a}$, we will employ the so-called perturbed test function method. That is nothing else than employing the multiscale ansatz (2.14) and making sure that products of weakly convergent sequences converge weakly, which is in general only possible under certain additional assumptions. These assumptions are given in the div-curl lemma. Below we sometimes use the shorthand notation $\int_U f := \int_U f(x)dx$.

Lemma 2.18. *(Div-curl lemma) Let* U *be a bounded, open. Assume that* $(p^\varepsilon) \subset L^2(U;\mathbb{R}^d)$ *and* $(V^\varepsilon) \subset L^2_{pot}(U) = \{V \in L^2(U;\mathbb{R}^d) \mid V = \nabla v \text{ for } v \in H_0^1(U)\}$ *satisfy, as* $\varepsilon \to 0$,

$$p^\varepsilon \rightharpoonup p_0 \text{ weakly in } L^2(U;\mathbb{R}^d) \text{ and } V^\varepsilon \rightharpoonup V_0 \text{ weakly in } L^2_{pot}(U).$$

Assume additionally that

$$\nabla \cdot p^\varepsilon \to f \text{ strongly in } H^{-1}(U).$$

Then the convergence of the product follows, in the sense that for all $\psi \in C_c^\infty(U)$:

$$\int_U p^\varepsilon \cdot V^\varepsilon \psi \to \int_U p_0 \cdot V_0 \psi.$$

Proof. Without loss of generality we may assume that $p_0 = V_0 = 0$ and $f = 0$. We have that by integration by parts with $V^\varepsilon = \nabla v^\varepsilon$ for $v^\varepsilon \in H_0^1(U)$,

$$\int_U p^\varepsilon \cdot V^\varepsilon \psi = \int_U p^\varepsilon \cdot \nabla v^\varepsilon \psi = \int_U \nabla \cdot p^\varepsilon v^\varepsilon \psi + \int_U v^\varepsilon p^\varepsilon \cdot \nabla \psi.$$

Since (V^ε) converges weakly to 0 in $L^2_{pot}(U)$, by Poincaré inequality (v^ε) is bounded in $H_0^1(U)$. By the compact Sobolev embedding thus (v^ε) converges strongly in $L^2(U)$ along subsequences. Every limit v needs to be equal, since $\nabla v = 0$ and v vanishes at the boundary. Thus the whole sequence converges strongly in $L^2(U)$ to v. Hence, together with the weak convergence of (p^ε) to 0 in $L^2(U)$, the product $(p^\varepsilon v^\varepsilon)$ converges weakly in $L^2(U)$ to 0. This shows that the second integral vanishes. To see that the first integral vanishes, we use that $(\nabla \cdot p^\varepsilon)$ strongly converges to zero in $H^{-1}(U)$ and that (v^ε) is bounded in $H^1(U)$. □

We can finally prove the stochastic homogenization result.

Theorem 2.19. *Let U be a bounded, open domain. Let Assumption 2.3 hold and let Ψ be given by Proposition 2.10, ϕ be given by Theorem 2.13 and $\bar{a}$ be given as in* (2.22). *Let $f \in L^2(U)$ and let u^ε solve*

$$\nabla \cdot (a(x/\varepsilon, \omega)\nabla u^\varepsilon) = f, \quad u^\varepsilon = 0 \text{ at } \partial U$$

and $\bar{u}$ solve the effective equation

$$\nabla \cdot (\bar{a}\nabla \bar{u}) = f, \quad \bar{u} = 0 \text{ at } \partial U.$$

Then, almost surely as $\varepsilon \to 0$

$$u^\varepsilon \rightharpoonup \bar{u} \text{ weakly in } H_0^1(U).$$

In particular, almost surely as $\varepsilon \to 0$

$$u^\varepsilon \to \bar{u} \text{ strongly in } L^2(U).$$

Proof. Due to the bound (2.11), we obtain that almost surely along a subsequence ε',

$$u^{\varepsilon'} \rightharpoonup v$$

weakly in $H_0^1(U)$ for some limit v with $v \in H_0^1(U)$ almost surely. We prove that v solves the effective equation, which by uniqueness proves that $v = \bar{u}$ and thus the convergence along the whole sequence follows. For simplicity denote the subsequence ε' again by ε. Let $\psi \in C_c^\infty(U)$. We define the perturbed test function

$$\psi^\varepsilon = \psi + \varepsilon\phi_i^t(x/\varepsilon, \omega)\partial_i\psi,$$

where ϕ^t solves the corrector equation for the transpose a^t according to Theorem 2.13. Testing the equation for u^ε by ψ^ε we obtain

$$\begin{aligned}
&\int_U a(x/\varepsilon, \omega)\nabla u^\varepsilon \cdot \nabla\psi^\varepsilon \\
&\quad = \int_U a(x/\varepsilon, \omega)\nabla u^\varepsilon \cdot (\nabla\phi_i^t(x/\varepsilon, \omega) + e_i)\partial_i\psi + \int_U a(x/\varepsilon, \omega)\nabla u^\varepsilon \cdot \nabla(\partial_i\psi)\varepsilon\phi_i^t(x/\varepsilon, \omega) \\
&\quad = \int_U f(\psi + \varepsilon\phi^t(x/\varepsilon, \omega)\partial_i\psi).
\end{aligned}$$

After transposing, we obtain by the ergodic theorem, Corollary 2.6, that almost surely

$$a^t(x/\varepsilon, \cdot)(\nabla\phi_i^t(x/\varepsilon, \cdot) + e_i) \rightharpoonup \bar{a^t} e_i \text{ weakly in } L^2_{loc}(U; \mathbb{R}^d).$$

Moreover, we have that $\nabla \cdot (a^t(x/\varepsilon, \cdot)(\nabla\phi_i^t(x/\varepsilon, \cdot) + e_i)) = 0$, since ϕ^t solves the cell equation for a^t and that almost surely $\nabla u^\varepsilon \rightharpoonup \nabla v$ weakly in $L^2(U; \mathbb{R}^d)$. Thus we can apply the div-curl lemma, Lemma 2.18, on that one-set and use $\bar{a^t} = (\bar{a})^t$ by Proposition 2.17 to obtain that almost surely as $\varepsilon \to 0$

$$\int_U a(x/\varepsilon,\omega)\nabla u^\varepsilon \cdot (\nabla\phi_i^t(x/\varepsilon,\omega)+e_i)\partial_i\psi \to \int_U \nabla v\cdot(\bar a)^t\nabla\psi = \int_U \bar a\nabla v\cdot\nabla\psi.$$

Furthermore, if we denote B the ball that satisfies $supp(\psi)\subset B$, we can bound

$$\left|\int_U a(x/\varepsilon,\cdot)\nabla u^\varepsilon\cdot\nabla(\partial_i\psi)\varepsilon\phi^t(x/\varepsilon,\cdot)\right| \le \Lambda\|\nabla(\partial_i\psi)\|_{L^\infty}\|\nabla u^\varepsilon\|_{L^2(U;\mathbb{R}^d)}\left(\int_B|\varepsilon\phi^t(x/\varepsilon,\cdot)|^2\right)^{1/2}$$

and

$$\left|\int_U f\varepsilon\phi^t(x/\varepsilon,\cdot)\partial_i\psi\right| \le \|\partial_i\psi\|_{L^\infty}\|f\|_{L^2(U)}\left(\int_B|\varepsilon\phi^t(x/\varepsilon,\cdot)|^2\right)^{1/2}.$$

Since (∇u^ε) is bounded in $L^2(U;\mathbb{R}^d)$ and ϕ^t is sublinear in the sense of Proposition 2.14 (applied to a^t), we obtain that those integrals vanish almost surely and in the limit we recover

$$\int_U \bar a\nabla v\cdot\nabla\psi = \int_U f\psi.$$

Hence, we obtain that v solves the effective equation. □

Remark 2.20. *(Periodic setting) In the case of periodic coefficient a, the proof is very similar, but the convergence of the perturbed test function can be seen more easily. Indeed, the sublinearity of ϕ^t follows immediately from the periodicity of the solution ϕ^t of the cell equation (which can be solved on the bounded domain $\mathbb{T}^d$ by Lax-Milgram), since for a 1-periodic function f it holds that $f(\cdot/\varepsilon) \rightharpoonup \int_0^1 f(x)dx = \langle f\rangle$ in $L^2_{loc}(\mathbb{R}^d)$, which can be readily seen by testing against indicator functions of intervals $[a,b]$.*

Notice that by the divergence theorem for $x\in U$, $r>0$ with $B_r(x)\subset U$,

$$\int_{\partial B_r(x)} a(y/\varepsilon,\omega)\nabla u^\varepsilon(y)\cdot\nu dS(y) + \int_{B_r(x)} f(y)dy = 0.$$

Thus we might also be interested in the convergence of the flux $a(\cdot/\varepsilon,\omega)\nabla u^\varepsilon$.

Theorem 2.21. *Let the assumptions of Theorem 2.19 hold. Then almost surely as $\varepsilon\to 0$,*

$$a(x/\varepsilon,\cdot)\nabla u^\varepsilon \rightharpoonup \bar a\nabla\bar u \quad \text{weakly in } L^2(U;\mathbb{R}^d).$$

Proof. Due to the upper bound on a, we have that

$$\|a(x/\varepsilon,\cdot)\nabla u^\varepsilon\|_{L^2(U)} \le \Lambda\|\nabla u^\varepsilon\|_{L^2(U)}$$

and the right-hand side is bounded almost surely by (2.11). Hence, almost surely $(a(x/\varepsilon,\cdot)\nabla u^\varepsilon)$ converges weakly to a limit w in $L^2(U;\mathbb{R}^d)$ along subsequences. Furthermore, due to the equation, we know that

$$\int_U w\nabla\psi = \int_U f\psi$$

for any $\psi \in C_c^\infty$. We prove that almost surely, for any $\xi \in \mathbb{R}^d$ and $\psi \in C_c^\infty$,

$$\int_U w \cdot \xi\psi = \int_U \bar{a}\nabla\bar{u} \cdot \xi\psi, \tag{2.23}$$

such that $w = \bar{a}\nabla\bar{u}$ follows. Due to the limit of each subsequence being unique, we conclude on convergence of the whole sequence. To show (2.23), we let $\psi \in C_c^\infty$ and define the perturbed test function

$$\psi_\xi^\varepsilon(x) = \xi \cdot x\psi(x) + \varepsilon\phi_\xi^t(x/\varepsilon,\omega)\psi(x)$$

for $\phi_\xi^t = \xi_i\phi_i^t$. To see that this is a good choice, notice that almost surely $\psi_\xi^\varepsilon \to \xi \cdot x\psi(x)$ strongly in $L^2(U)$ due to the sublinearity of ϕ^t from Proposition 2.14 and thus

$$\int_U f\psi_\xi^\varepsilon \to \int_U f\xi \cdot x\psi = \int_U w \cdot \xi\psi + \int_U w \cdot \nabla\psi(\xi \cdot x). \tag{2.24}$$

Now testing the equation of u^ε by the perturbed test function yields

$$\int_U a(x/\varepsilon,\cdot)\nabla u^\varepsilon \cdot (\nabla\phi_\xi^t(x/\varepsilon,\cdot)+\xi)\psi + \int_U a(x/\varepsilon,\cdot)\nabla u^\varepsilon \cdot \nabla\psi(\xi \cdot x + \varepsilon\phi_\xi^t(x/\varepsilon,\cdot)) = \int_U f\psi_\xi^\varepsilon. \tag{2.25}$$

After transposing a, we obtain $a^t(x/\varepsilon,\cdot)(\nabla\phi_\xi^t(x/\varepsilon,\cdot)+\xi) \rightharpoonup (\bar{a})^t\xi$ in $L^2_{loc}(U)$ by Corollary 2.6. Moreover, $\nabla u^\varepsilon \rightharpoonup \nabla\bar{u}$ in $L^2(U)$ by Theorem 2.19 and ϕ^t solves the cell equation for a^t. Thus we can apply the div-curl lemma to see that

$$\int_U a(x/\varepsilon,\cdot)\nabla u^\varepsilon \cdot (\nabla\phi_\xi^t(x/\varepsilon,\cdot)+\xi)\psi \to \int_U \bar{a}\nabla\bar{u} \cdot \xi\psi.$$

For the second integral on the left-hand side of (2.25), we use that $a(x/\varepsilon,\cdot)\nabla u^\varepsilon \rightharpoonup w$ in $L^2(U)$ and the sublinearity of ϕ^t from Proposition 2.14 to obtain that

$$\int_U a(x/\varepsilon,\cdot)\nabla u^\varepsilon \cdot \nabla\psi(\xi \cdot x + \varepsilon\phi_\xi^t(x/\varepsilon,\cdot)) \to \int_U w\nabla\psi(\xi \cdot x).$$

Together with (2.24), (2.23) follows. □

2.11. **Strong convergence of the two-scale expansion.**

Theorem 2.22. *Let $f \in C^\alpha(U)$ for a bounded, open, connected $C^{2,\alpha}$-domain U and $\alpha \in (0,1)$. Let the assumptions of Theorem 2.19 hold. Then, almost surely as $\varepsilon \to 0$*

$$u^\varepsilon - \bar{u} - \varepsilon\phi_i(x/\varepsilon,\cdot)\partial_i\bar{u} \to 0 \text{ strongly in } H^1(U).$$

Proof. Let

$$w^\varepsilon := u^\varepsilon - \bar{u} - \varepsilon\phi_i(x/\varepsilon,\cdot)\partial_i\bar{u}.$$

From Theorem 2.19 we know that almost surely $u^\varepsilon \to \bar{u}$ and from Proposition 2.14, that $\varepsilon\phi_i(x/\varepsilon,\omega) \to 0$ strongly in $L^2(U)$. Thus we obtain that almost surely $w^\varepsilon \to 0$ strongly in $L^2(U)$. We are left to prove that almost surely $\nabla w^\varepsilon \to 0$ strongly in $L^2(U)$. To that aim, we define for any $\rho \in (0,1)$,

$$w^{\varepsilon,\rho} := u^\varepsilon - \bar{u} - \varepsilon\phi_i(x/\varepsilon,\cdot)\eta_\rho\partial_i\bar{u}$$

for a smooth cutoff function $\eta_\rho : \mathbb{R}^d \to [0,1]$, that satisfies $\eta_\rho(x) = 1$ if $d(x,\partial U) \geq \rho$ and $\eta_\rho(x) = 0$ if $d(x,\partial U) < \rho/2$ and $|\nabla\eta_\rho| \leq C/\rho$ for a constant $C \in (0,\infty)$. Define the flux

$$q_i = a(e_i + \nabla\phi_i),$$

which is divergence-free by the equation for ϕ_i, that is $\nabla \cdot q_i = 0$. Then we obtain using the equations for $u^\varepsilon, \bar{u}$ and $\bar{a}e_i = \mathbb{E}[q_i]$ that

$$\begin{aligned}
-\nabla\cdot(a(x/\varepsilon,\cdot)\nabla w^{\varepsilon,\rho}) &= \nabla\cdot[(a(x/\varepsilon,\cdot)-\bar{a})\nabla\bar{u}] + \nabla\cdot(a(x/\varepsilon,\cdot)\nabla[\phi_i(x/\varepsilon,\cdot)](\eta_\rho\partial_i\bar{u})) \\
&\quad + \nabla\cdot(a(x/\varepsilon,\cdot)\varepsilon\phi_i(x/\varepsilon,\cdot)\nabla(\eta_\rho\partial_i\bar{u})) \\
&= \nabla\cdot[(1-\eta_\rho)(a(x/\varepsilon,\cdot)-\bar{a})\nabla\bar{u}] + \nabla\cdot(\varepsilon(q_i(x/\varepsilon,\cdot)-\mathbb{E}[q_i])(\eta_\rho\partial_i\bar{u})) \\
&\quad + \nabla\cdot(a(x/\varepsilon,\cdot)\varepsilon\phi_i(x/\varepsilon,\cdot)\nabla(\eta_\rho\partial_i\bar{u})).
\end{aligned}$$

Testing against $w^{\varepsilon,\rho}$ and using that $w^{\varepsilon,\rho}$ vanishes at the boundary ∂U, we obtain by the uniform ellipticity of a and Hölder's inequality,

$$\begin{aligned}
\|\nabla w^{\varepsilon,\rho}\|^2_{L^2(U)} &= \int_U |\nabla w^{\varepsilon,\rho}|^2 \\
&\leq \lambda^{-1}\int_U a(x/\varepsilon,\cdot)\nabla w^{\varepsilon,\rho}\cdot\nabla w^{\varepsilon,\rho} \\
&= \lambda^{-1}\int_U -\nabla\cdot[a(x/\varepsilon,\cdot)\nabla w^{\varepsilon,\rho}]w^{\varepsilon,\rho} \\
&\leq \lambda^{-1}\|\nabla w^{\varepsilon,\rho}\|_{L^2(U)}(\|1-\eta_\rho\|_{L^2(U)}\Lambda\|\bar{u}\|_{C^{2,\alpha}(U)} + \|\partial_i\bar{u}\|_{L^2(U)}\|\varepsilon(q_i(\cdot/\varepsilon,\cdot)-\mathbb{E}[q_i])\|_{L^2(U)} \\
&\qquad + (C\rho^{-1}\Lambda + \Lambda\|\partial_i\bar{u}\|_{L^2(U)})\|\varepsilon\phi_i(\cdot/\varepsilon,\cdot)\|_{L^2(U)}).
\end{aligned}$$

Due to almost sure weak convergence of $(q(\cdot/\varepsilon,\cdot))$ in $L^2_{loc}(\mathbb{R}^d)$ by Corollary 2.6 and sublinearity of $(\phi_i(\cdot/\varepsilon,\cdot))$ by Proposition 2.14, we obtain that almost surely

$$\|\varepsilon(q_i(\cdot/\varepsilon,\cdot)-\mathbb{E}[q_i])\|_{L^2(\bar{U})} + \|\varepsilon\phi_i(\cdot/\varepsilon,\cdot)\|_{L^2(\bar{U})} \to 0.$$

Hence we have

$$\limsup_{\varepsilon\to 0}\|\nabla w^{\varepsilon,\rho}\|_{L^2(U)} \leq \lambda^{-1}\Lambda\|1-\eta_\rho\|_{L^2(U)}\|\bar{u}\|_{C^{2,\alpha}(U)} \leq c\lambda^{-1}\Lambda\|1-\eta_\rho\|_{L^2(U)}\|f\|_{C^\alpha(U)}, \tag{2.26}$$

where the last inequality follows from the regularity of the domain for some constant $c \in (0,\infty)$. Writing

$$\nabla w^\varepsilon = \nabla w^{\varepsilon,\rho} + \varepsilon\phi_i(x/\varepsilon,\cdot)\nabla[(1-\eta_\rho)\partial_i\bar{u}] + \nabla\phi_i(x/\varepsilon,\cdot)(1-\eta_\rho)\partial_i\bar{u}$$

we obtain that by the ergodic theorem, Corollary 2.6, and (2.26) for a possibly different constant $c \in (0, \infty)$,

$$\limsup_{\varepsilon \to 0} \|\nabla w^\varepsilon\|_{L^2(U)} \leq c\lambda^{-1}\Lambda \|1 - \eta_\rho\|_{L^2(U)} \|f\|_{C^\alpha(U)} \left[1 + \sum_{i=1}^d \mathbb{E}[|\nabla \phi_i|^2]\right].$$

Since $\|1 - \eta_\rho\|_{L^2(U)} \to 0$ for $\rho \to 0$ and $\rho \in (0, 1)$ was arbitrary, the claim follows. □

Remark 2.23. *(Uniform ellipticity assumption) It is possible to relax the lower bound in the uniform ellipticity assumption assuming that $\lambda(\omega)$ is a random variable which is positive (but not necessarily uniformly in ω) and satisfies $\mathbb{E}[\lambda^{-d}] < \infty$. In the limit $\varepsilon \to 0$ the equation then becomes uniformly elliptic due to the moment bounds on λ^{-1}, cf.* [2].

Remark 2.24. *(Non-divergence form equations) In the periodic case, it is possible to consider non-divergence form operators $\mathscr{L} = a_{i,j}\partial_i\partial_j + b_i\partial_i$ on $\mathbb{T}^d$ and derive the homogenization result. The difference is essentially, that there will be a second corrector equation, which is solvable due to the Fredholm alternative. However, in the stochastic case, the solvability of that second corrector equation is in general unclear. Only in certain special cases $b = 0$, b divergence-free or $b = \nabla\tilde{b}$ for stationary $\tilde{b}$, it is known to be solvable (cf. also the book* [6]*). For more details, we refer to* [4, section 4] *and the references therein.*

3. Stochastic homogenization for nonlinear divergence-form equations

In this section, we generalize the previous results to quasilinear elliptic equations. The reference for this section is the book [9]. For the sake of presentation, we only consider here the case of single-valued monotone operators in divergence-form, that do not contain the lower order component a_0 without the divergence in front, while the book [9] contains more general cases (pseudomonotone and also multi-valued operators). Some of the ideas are similar as in the previous section on linear equations, so we keep them short in this section. Moreover comparing to the last section, in this section a lot of difficulties in proving the homogenization result are outsourced in the compactness Theorem 3.9 below, while in the previous section, we proved a lot directly "by hand".

We consider the following nonlinear boundary-value problem

$$-\nabla \cdot a(x, \omega, u, \nabla u) = f \text{ in } U, \quad u = 0 \text{ at } \partial U \tag{3.1}$$

with $u \in V = W^{1,p}(U)$ for $U \subset \mathbb{R}^d$ open, bounded and $f \in V^* = W^{-1,p'}(U)$ with p' conjugate to $p \in (1, \infty)$ and $a : \mathbb{R}^d \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, $a(x, \omega, u, v)$, which is assumed to be measurable in the variables x, ω and continuous in (u, v). The corresponding nonlinear operator $\mathscr{A}(\omega)$ is defined as

$$\mathscr{A}(\omega) : V \to V^*, \quad \mathscr{A}(\omega)(u) = -\nabla \cdot a(x, \omega, u, \nabla u), \tag{3.2}$$

which is welldefined under a boundedness assumption on a, see below (n1).

3.1. Assumptions, ergodicity and wellposedness.

Assumption 3.1. *(Random environment)*[1] *The assumptions on the probability space* $(\Omega, \mathscr{F}, \mathbb{P})$ *equipped with the measure-preserving transformation* (τ_x) *remain the same as in Assumption 2.3 (i.e. stationarity (2), ergodicity (3)). The uniform ellipticity assumption, as well as the continuity in probability assumption will be replaced by the following set of assumptions. We assume that there exist constants* $\kappa > 0$, $c_1, c_2, h, \theta \geq 0$, $s \in (0, \min(p, p')]$ *and a modulus of continuity* $\nu(r)$*, such that*

(n1) *Boundedness: almost surely for any* $(x, u, v) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$,

$$|a(x, \omega, u, v)|^{p'} \leq c_1 + c_2(|u|^p + |v|^p)$$

(n2) *Strict monotonicity: almost surely for any* $(x, u_1, v_1), (x, u_2, v_2) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$,

$$[a(x, \omega, u_1, v_1) - a(x, \omega, u_2, v_2)] \cdot (v_1 - v_2) \geq \kappa |v_1 - v_2|^p$$

(n3) *Continuity: almost surely for any* $(x, u_1, v_1), (x, u_2, v_2) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$,

$$\begin{aligned} &|a(x, \omega, u_1, v_1) - a(x, \omega, u_2, v_2)|^{p'} \\ &\quad \leq \theta[(h + |v_1|^p + |v_2|^p)\nu(|u_1 - u_2|) + (h + |v_1|^p + |v_2|^p)^{1-s/p}|v_1 - v_2|^s]. \end{aligned}$$

We note that (n1) and (n2) together imply

(n4) Coercivity: almost surely for any $(x, u, v) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$,

$$a(x, \omega, u, v) \cdot v \geq \kappa |v|^p - c_1.$$

Definition 3.2. *We call* E_U^1 *the class of operators of the form*

$$\mathscr{A} : V = W^{1,p}(U) \to V^*, \quad \mathscr{A}(u) = -\nabla \cdot a(x, u, \nabla u)$$

with $x \mapsto a(x, u, v)$ *measurable and* $(u, v) \mapsto a(x, u, v)$ *continuous that satisfy (n1),(n2) and* E_U^2 *the class of operators of that kind with coefficient* a *that satisfy (n1) - (n3).*

For a Banach space V, let V^* be its dual and for $U \in V^*$ and $w \in V$ let $\langle U, w \rangle_{V^*,V}$ be the dual pairing.
Under the assumption (n1), $\mathscr{A}(\omega)$ is a well-defined. Moreover $\mathscr{A}(\omega)$ is strictly monotone due to (n2), that is

$$\langle \mathscr{A}(\omega)(u) - \mathscr{A}(\omega)(v), u - v \rangle_{V^*,V} > 0 \text{ if and only if } u \neq v$$

and coercive due to (n4) and $p > 1$, that is

$$\lim_{\|u\|_V \to \infty} \frac{\langle \mathscr{A}(\omega)(u), u \rangle_{V^*,V}}{\|u\|_V} = \infty,$$

as well as bounded due to (n1) (i.e. maps bounded sets into bounded sets) and hemicontinuous due to (n3), that is for all $u, v, w \in V$ the map $[0, 1] \ni t \to \langle \mathscr{A}(\omega)(u+tv), w \rangle_{V^*,V}$ is continuous.

[1][9, assumptions in section 3.2.4]

Example 3.3. *An example satisfying the assumptions (n1)-(n3) is the stationary p-Laplace equation where $a(x,u,v) = a(x,v) = b(x)|v|^{p-2}v$ for $p \geq 2$, where b is a bounded, measurable function.*

We now give the nonlinear analogon to the Lax-Milgram lemma.

Theorem 3.4. *(Minty-Browder) Let V be a real reflexive Banach space and $A : V \to V^*$ be a map, that is monotone, coercive and hemicontinuous. Then A is surjective. If A is moreover strictly monotone, then A is bijective.*

As a corollary from the Minty-Browder theorem, we obtain the wellposedness of the nonlinear equation (cf. [12, Theorem 10.51]).

Proposition 3.5. *(Wellposedness) Let $\mathscr{A}$ be given as in (3.2) with coefficient a satisfying (n1)-(n3). Then for any $f \in W^{-1,p'}(U)$, almost surely there exists a unique solution $u \in W^{1,p}(U)$ to the problem (3.1).*

3.2. **Homogenization and Gamma convergence.**
Homogenization for nonlinear problems is typically studied as a problem of convergence of the operators $(\mathscr{A}^\varepsilon)$ for

$$\mathscr{A}^\varepsilon(\omega)(u) = \mathscr{A}(\tau_{x/\varepsilon}\omega)(u) = \nabla \cdot a(x/\varepsilon, \omega, u, \nabla u), \quad u \in V \tag{3.3}$$

towards a deterministic operator $\bar{\mathscr{A}}$ in a certain class of operators. The convergence type, which turns out to be handy is the strong G-convergence.

Definition 3.6. *(Strong G-convergence for operators in E^1_U)[2] Let $\mathscr{A}^k(u) = \nabla \cdot a^k(x,u,\nabla u)$ for $k \in \mathbb{N}$ and $\mathscr{A}(u) = \nabla \cdot a(x,u,\nabla u)$ be of class E^1_U. Set for $u_1, u_2 \in V$,*

$$\mathscr{A}(u_1,u_2) := \nabla \cdot a(x,u_1,\nabla u_2), \quad \mathscr{A}^k(u_1,u_2) := \nabla \cdot a^k(x,u_1,\nabla u_2),$$

such that $\mathscr{A}(u,u) = \mathscr{A}(u)$, $\mathscr{A}^k(u,u) = \mathscr{A}^k(u)$. Let for $u_1, u_2 \in V$, u^k be the solution of

$$\mathscr{A}^k(u_1,u^k) = \mathscr{A}(u_1,u_2).$$

Then set for $u_1, u_2 \in V$,

$$\Gamma(u_1,u_2) = a(x,u_1,\nabla u_2) \quad \text{and} \quad \Gamma^k(u_1,u_2) = a^k(x,u_1,\nabla u^k).$$

We say that $(\mathscr{A}^k)$ strongly G-converges to $\mathscr{A}$, denoted by $\mathscr{A}^k \Rightarrow \mathscr{A}$, if for any $u_1, u_2 \in V$

- *$u^k \rightharpoonup u_2$ weakly in V and*
- *$\Gamma^k(u_1,u_2) \rightharpoonup \Gamma(u_1,u_2)$ weakly in $(L^{p'}(U))^d$.*

Remark 3.7. *We have that $\mathscr{A}^k \Rightarrow \mathscr{A}$ implies $\mathscr{A}^k(u_1,\cdot) \Rightarrow \mathscr{A}(u_1,\cdot)$ for any $u_1 \in V$. In particular, the strong G-limit is unique (*[9, Proposition 2.3.1]*). Strong G-convergence to $\mathscr{A}$*

[2] [9, Definition 2.3.1]

holds if and only if for any subsequence there exists a subsequence, that strongly G-converges to $\mathscr{A}$.

Remark 3.8. *Notice that if $\mathscr{A}^k \Rightarrow \mathscr{A}$ and $a^k(x,u,v) = a^k(x,v)$, $a(x,u,v) = a(x,v)$ do not depend on the variable $u \in \mathbb{R}$ for every k, it follows that for u^k solving $\mathscr{A}^k(u^k) = f$ and u solving $\mathscr{A}(u) = f$, $u^k \rightharpoonup u$ weakly in $V = W^{1,p}(U)$ and for the flux $a^k(x,\nabla u^k) \rightharpoonup a(x,\nabla u)$ weakly in $(L^{p'}(U))^d$. This motivates the usage of Gamma convergence in homogenization theory, since these are the convergences we expect to have (see in the last section).*

The following compactness theorem we take from [9, Theorem 2.3.1, Lemma 2.3.2] without proof. Essentially it follows from a very general compactness theorem for Gamma convergent operators ([9, Proposition 1.1.4 and 1.2.3]). Notice however that for operators in the class E_U^2, it is possible to have the limit $\bar{\mathscr{A}} \in E_U^2$ (while the analogue compactness statement in E_U^1 is not true).

Theorem 3.9. *(G-compactness of E_U^2 operators) For any sequence $(\mathscr{A}^k)_k$ in E_U^2, there exists a subsequence that converges to an operator $\bar{\mathscr{A}} \in E_U^2$ and satisfies the bounds with explicit constants $\bar{c}_1, \bar{c}_2, \bar{\kappa}, \bar{h}, \bar{\theta}, \bar{s}$ and modulus of continuity $\bar{\nu}(r)$.*

We give an alternative characterization of Gamma-convergence in E_U^2, that turns out to be very useful in homogenization theory. The next proposition is [9, Theorem 2.4.1].

Proposition 3.10. *Assume that $\mathscr{A}^k \in E_U^2$, $k \in \mathbb{N}$, and that for any $\xi = (u,v) \in \mathbb{R}\times\mathbb{R}^d$ there exists a sequence (w^k) such that $w^k \rightharpoonup v \cdot x$ weakly in $\bar{V}$, $(a^k(x,u,\nabla w^k))$ is weakly convergent in $L^{p'}(U)^d$ and $(\nabla \cdot a^k(x,u,\nabla w^k))$ is precompact in V^*. Then $(\mathscr{A}^k)$ is strongly G-convergent to a limit operator $\mathscr{A}$ and $a^k(x,u,\nabla w^k) \rightharpoonup a(x,u,v)$ weakly in $L^{p'}(U)^d$.*

The converse statement is also true and can be easily seen by choosing w^k solving $\mathscr{A}^k(u,w^k) = \mathscr{A}(u, v\cdot x)$.

3.3. **The random homogenization corrector.**

Like in the last section, we solve the equation for the gradient of the corrector on the probability space and identify $a(x,\omega,\cdot)$ with $a(0,\tau_x\omega,\cdot) = a(\tau_x\omega,\cdot)$. Similar, we define for $p \in (1,\infty)$, $L^p_{pot}(\Omega) = \overline{\{F \in L^p(\Omega,\mathbb{R}^d) \mid F = Df \text{ for } f \in \mathscr{H}^1_p\}}^{L^p}$, where $\mathscr{H}^1_p$ is the Banach space with norm $\|f\|_{\mathscr{H}^1_p} := \|f\|_{L^p(\Omega)} + \|Df\|_{L^p(\Omega)}$. Then we have $L^p_{pot}(\Omega)^\perp = L^{p'}_{sol}(\Omega)$ with respect to the dual pairing.

Proposition 3.11. *(Equation for the gradient of the corrector) For any $\xi = (u,v) \in \mathbb{R}\times\mathbb{R}^d$, there exists a unique solution $\chi_\xi \in L^p_{pot}(\Omega)$ to*

$$\mathbb{E}[a(\cdot,u,v+\chi_\xi)\cdot\varphi] = 0 \textit{ for all } \varphi \in L^p_{pot}(\Omega).$$

Proof. Define the map $\mathscr{B}_\xi : L^p_{pot}(\Omega) \to (L^p_{pot}(\Omega))^*$ with $\mathscr{B}_\xi(\chi)\varphi = \langle a(\cdot,u,v+\chi),\varphi\rangle_{(L^p_{pot})^*,L^p_{pot}}$ for $\chi,\varphi \in L^p_{pot}$. Notice that $(L^p_{pot})^*$ can be identified with $L^{p'}(\Omega)/L^{p'}_{sol}(\Omega)$ and for any

$\psi \in L^{p'}_{sol}(\Omega)$, $D\cdot(a(\cdot,u,v+\chi)+\psi) = D\cdot a(\cdot,u,v+\chi)$ and hence $a(\cdot,u,v+\chi) \in L^{p'}(\Omega)/L^{p'}_{sol}(\Omega)$. Clearly if $\mathscr{B}(\chi_\xi) = 0$, then χ_ξ solves the equation. We have that $\mathscr{B}_\xi$ is a bounded, continuous, strictly monotone and coercive operator due to assumptions (n1)-(n3). Thus by Minty-Browder, there exists a unique χ_ξ with $\mathscr{B}_\xi(\chi_\xi) = 0$. □

3.4. The homogenized coefficient.

We define the homogenized coefficient by

$$\bar{a}(u,v) = \mathbb{E}[a(\cdot,u,v+\chi_\xi)], \quad \xi = (u,v) \in \mathbb{R}\times\mathbb{R}^d \tag{3.4}$$

and the homogenized operator by

$$\bar{\mathscr{A}}(u) = \nabla\cdot\bar{a}(u,\nabla u), \quad u \in V. \tag{3.5}$$

It is not easy to verify by hand that $\bar{\mathscr{A}} \in E^2_U$ and thus that the homogenized equation

$$\nabla\cdot\bar{a}(u,\nabla u) = f, \quad u = 0 \text{ at } \partial U$$

is wellposed. However, the compactness theorem yields that the limit operator is of class E^2_U.

3.5. Proof of the homogenization result.

Combining the above, we can conclude on the homogenization result, that is [9, Theorem 3.2.2, section 3.2.4].

Theorem 3.12. *(Homogenization result) Let $\mathscr{A} \in E^2_U$ be given as in* (3.2) *and let for $\varepsilon \in (0,1)$, $\mathscr{A}^\varepsilon$ be as in* (3.3)*. Then it follows that almost surely $\mathscr{A}^\varepsilon \Rightarrow \bar{\mathscr{A}}$ and $\bar{\mathscr{A}} \in E^2_U$. In particular, if $a^k(x,\omega,u,v) = a^k(x,\omega,v)$ does not depend on the variable $u \in \mathbb{R}$ for every k, then the solutions satisfy almost surely*

$$u^k \rightharpoonup \bar{u} \text{ weakly in } W^{1,p}(U), \quad a^k(x,\cdot,\nabla u^k) \rightharpoonup \bar{a}(x,\nabla\bar{u}) \text{ weakly in } L^{p'}(U)^d.$$

Proof. By Theorem 3.9 there exists a subsequence $(\mathscr{A}^{\varepsilon'})$ that converges to $\tilde{\mathscr{A}} \in E^2_U$ with coefficient $\tilde{a}$. Due to Remark 3.7, it thus suffices to prove that $\bar{a} = \tilde{a}$. Let $\xi = (u,v) \in \mathbb{R}\times\mathbb{R}^d$. To simplify notation denote ε' by ε. Let

$$\chi(x,\omega) = \chi_\xi(\tau_x\omega), \quad \chi^\varepsilon(x,\omega) = \chi_\xi(\tau_{x/\varepsilon}\omega),$$

where χ_ξ is given from Proposition 3.11. By the ergodic theorem, Corollary 2.6, we have that almost surely $\chi^\varepsilon \rightharpoonup 0 = \mathbb{E}[\chi_\xi]$ weakly in $L^p_{loc}(\mathbb{R}^d,\mathbb{R}^d)$. Similar as before (cf. Proposition 2.12), we have that $\chi = \nabla N$ for $N \in W^{1,p}_{loc}(\mathbb{R}^d)$. Let $N^\varepsilon(x,\omega) = N(\tau_{x/\varepsilon}\omega)$ with $\nabla(\varepsilon N^\varepsilon) = \chi^\varepsilon$. Then by Poincaré inequality for a ball B, almost surely

$$\varepsilon(N^\varepsilon - \langle N^\varepsilon\rangle_B) = \varepsilon\left(N^\varepsilon - |B|^{-1}\int_B N^\varepsilon(x)dx\right) \to 0 \tag{3.6}$$

strongly in $L^p(B)$ (cf. Proposition 2.14). Define $w^\varepsilon = v\cdot x + \varepsilon(N^\varepsilon - \langle N^\varepsilon\rangle_B)$ for a ball B, such that $U \subset B$. Then it follows that $w^\varepsilon \to v\cdot x$ in $L^p(U)$ by the sublinearity of the corrector N

from (3.6) and $\nabla w^\varepsilon = v + \chi^\varepsilon \rightharpoonup v$ weakly in $L^p(U)^d$. Thus $w^\varepsilon \rightharpoonup v \cdot x$ in $W^{1,p}(U)$. By the ergodic theorem, Corollary 2.6, we obtain that almost surely

$$a(\tau_{x/\varepsilon}\cdot, u, \nabla w^\varepsilon) \rightharpoonup \mathbb{E}[a(\cdot, u, v + \chi_\xi)] = \bar{a}(u, v)$$

weakly in $L^{p'}_{loc}(\mathbb{R}^d)$ and by the equation for χ_ξ,

$$\nabla \cdot a(\tau_{x/\varepsilon}\cdot, u, \nabla w^\varepsilon) = 0.$$

Then Proposition 3.10 implies that $\tilde{a}(u, v) = \bar{a}(u, v)$. □

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