§1. Nef line bundle

X = complete variety or scheme

Def: A line bundle L on X is **numerically effective** or **nef** if

$$\int_C c_1(L) \geq 0,$$

$\forall$ irreducible curves $C \subseteq X$.

$\Updownarrow$

$$(L \cdot C) \geq 0$$

A Cartier divisor D on X (with $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ coefficients) is **nef** if $(D \cdot C) \geq 0$, $\forall$ irre. curve $C \subseteq X$

RMK: This defition only depends on the numerical equivalence class of L or D.

Ex:

① Ample line bundle L

$(L \cdot C) > 0$, $\forall C$

② Trivial line bundle $\mathcal{O}_X$

$c_1(\mathcal{O}_X) = 0$

$\Rightarrow (c_1(\mathcal{O}_X) \cdot C) = 0$, $\forall C$.

Def: A line bundle L on X is semi-ample if $L^{\otimes m}$ is globally generated for some $m > 0$

A divisor D is semi-ample if its corresponding line bundle

is semi-ample.

Def $\Rightarrow$ $\forall x \in X$, $\exists s \in H^0(X, L^{\otimes m})$ s.t. $s(x) \neq 0$

$\Rightarrow$ morphism $\varphi : X \longrightarrow \mathbb{P}^N$

$x \longmapsto [s_0(x) : \cdots : s_N(x)]$

Rmk: φ need not to be an embedding. In particular,

$\{\text{Ample}\} \subsetneq \{\text{Semi-ample}\} \subseteq \{\text{nef}\}$

$(\deg(L|_C) = \sum_{p \in C} \operatorname{ord}_p(s))$

Ex: ① Semi ample but not ample

$X = \mathbb{P}^1 \times \mathbb{P}^1$

$L = \mathcal{O}(0,1)$

↑

The pull back of $\mathcal{O}(1)$ along the projection $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$

$H^0(X, L) = \{ay_1 + by_2 \mid a, b \in k, [y_1 : y_2] \in \mathbb{P}^1\}$

L is basepoint free

Hence, L is semi-ample.

A curve in X is defined by a bihomogeneous polynomial of degree (m, n).

$O(0,1) \cdot (m,n) = 0 \cdot n + 1 \cdot m = m \geq 0$

$\Rightarrow L$ is nef. $\downarrow$ $D_1 = \mathbb{P}^1 \times \{p \in\}$

$D_2 = \{p \in\} \times \mathbb{P}^1$

$(0 \cdot D_1 + D_2) \cdot (mD_1 + nD_2)$

Consider the curve defined by $(0, n)$

$O(0,1) \cdot (0, n) = 0$

Not ample

② Effective divisor but not nef.

$X = Bl_p \mathbb{P}^2$

$L = E$ exceptional divisor

Since $E \cong \mathbb{P}^1$, it is an effective divisor.

but $E^2 = -1 < 0$ $\Rightarrow$ Not nef.

③ Nef but not semi-ample.

$X = Bl_q \mathbb{P}^2$, blow-up of $\mathbb{P}^2$ at q general point.

$L = -K_X$, anticanonical divisor

$= 3H - E_1 - \cdots - E_q$.

where H is the hyperplane class and E_i are exceptional divisors.

Property:

1) The pullback of nef line bundle along proper morphism is nef

2) If $\begin{cases} 1) & f: X \to Y \text{ surjective + proper} \\ 2) & f^*(L) \text{ is nef} \end{cases}$
then L is nef

3) L is nef on $X \Leftrightarrow L_{red}$ is nef on X_{red}.
$L_{red} = \pi^*(L)$, $\pi: X_{red} \to X$
$\mathcal{O}_{X_{red}} = \mathcal{O}_X / N \leftarrow$ Nivadical.

4) L is nef $\Leftrightarrow$ Its restriction to each irreducible component of X is nef.

5) If $\begin{cases} 1) & D \subseteq X \text{ effective divisor} \\ 2) & \text{The normal bundle } N_{D/X} \cong \mathcal{O}_D(D) \text{ of } D \text{ is nef} \end{cases}$
then D is nef

$\Downarrow$

If $\begin{cases} X = \text{surface} \\ C = \text{irre. curve} \end{cases}$
then $(C^2) \geq 0 \Rightarrow C$ is nef

6) If a connected algebraic group acting on X transitively, then any effective divisor D on X is nef.

§2 Kleiman's Thm

Statement:

1) D is a nef $\mathbb{R}$-divisor on X iff $(D^k \cdot V) \geq 0$, $\forall$ irre. variety $V \subseteq X$ of dim k.
2) $\forall$ nef line bundle L on X $\int_V c_1(L)^{\dim(V)} \geq 0$

Corollary: Let X be a proj. var. or sch.
D be a nef $\mathbb{R}$-divisor
H be a ample $\mathbb{R}$-divisor
Then $D + \epsilon H$ is ample, $\forall \epsilon > 0$

Conversely, if D, H are any two divisors s.t. $D + \epsilon H$ is ample for $0 < \epsilon \ll 1$, then D is nef.

$N^1(X)_{\mathbb{R}} :=$ The finite dimensional real v.s of numerically equivalence classes of Cartier divisors on X with $\mathbb{R}$-coefficients.

Def: The ample cone $\mathrm{Amp}(X)$ (Nef(X)) of X is the convex cone of all ample $\mathbb{R}$-divisor classes on X (Nef)

Restate: X is projective vari. or sch.

1) $Nef(X) = \overline{Amp(X)}$
2) $Amp(X) = int(Nef(X))$

Proof: $\begin{cases} Nef(X) \text{ is closed.} \\ Amp(X) \text{ is open.} \\ Amp(X) \subset Nef(X) \end{cases}$

$\Rightarrow \begin{cases} \overline{Amp(X)} \subseteq Nef(X), \\ Amp(X) \subseteq int(Nef(X)) \end{cases}$

Let H be ample.

$\forall$ any nef divisor D. $D + \epsilon H$ is ample $\forall \epsilon > 0$

So $D \in \overline{Amp(X)}$. (1) $\forall$

Let $D' \in int(Nef(X))$

Then for $0 < \epsilon \ll 1$, $D' - \epsilon H'$ is nef, $D' = (D' - \epsilon H') + \epsilon H'$ is ample.

($D' - \epsilon H'$: nef; $\epsilon H'$: ample.)

□

§3 Iitaka fibration

X proj. var.
L line bundle,
$Bs|L|$ base locus of L

$$\Downarrow$$

$$\varphi_{|L|} : X \setminus Bs|L| \longrightarrow \mathbb{P}(H^0(X, L))$$
$$x \mapsto [s_0(x) : \cdots : s_r(x)]$$

$$\Downarrow$$

$$\varphi : X \dashrightarrow \mathbb{P}^r$$

$N(L) := \{ m \in \mathbb{N} \mid H^0(X, L^{\otimes m}) \neq 0 \}$.
$\forall\, m \in N(L)$, we get

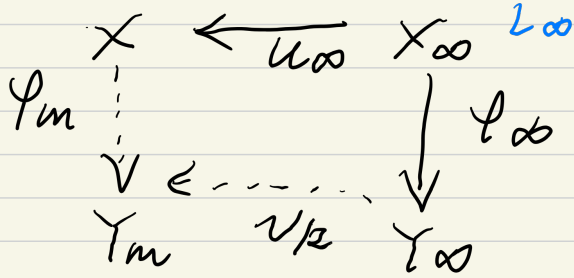
$$\varphi_{|L^{\otimes m}|} : X \dashrightarrow \mathbb{P}(H^0(X, L^{\otimes m}))$$
$$Y_m := \varphi_{|L^{\otimes m}|}(X)$$

The Iitaka dim of L is

$$\kappa(X, L) = \begin{cases} \max_m \{\dim(Y_m)\}, & N(L) \neq \{0\} \\ -\infty, & N(L) = \{0\} \end{cases}$$

Kodaira dim of X
= Iitaka dim of K_X

Thm (Iitaka fibration)
X normal projective variety
L line bundle on X s.t. $\kappa(X, L) > 0$
Then for sufficiently large $m \in N(L)$
the rational map $\varphi_m : X \dashrightarrow Y_m$
is birational equivalent to a fixed fibration $\varphi_\infty : X_\infty \to Y_\infty$ Iitaka fibration associated to L.
and $\dim(Y_\infty) = \kappa(X, L)$.



Moreover $L_\infty := u_\infty^*(L)$
$\kappa(F, L_\infty|_F) = 0$
$\uparrow$
$\kappa(F) = 0$