Taylor's Inequality

The remainder of the Taylor series for $f(x)$ centered at a is defined as

$$R_n(x) = f(x) - T_n(x),$$

where $T_n(x)$ denotes the nth degree Taylor polynomial (also centered at a). Taylor's Inequality says that if $\left|f^{(n+1)}(x)\right| \leq M$ for all x satisfying $|x-a| \leq d$, then

$$\left|R_n(x)\right| \leq \frac{M}{(n+1)!}\left|x-a\right|^{n+1} \text{ for } x \text{ satisfying } |x-a| \leq d.$$

1. Use Taylor's Inequality to determine what degree Taylor polynomial centered at $a = \frac{\pi}{6}$ is needed to estimate $\cos(\frac{1}{2})$ to within $0.0001 = 10^{-4}$.

If you're using a Taylor polynomial to approximate f at a particular value of x instead of a range of x, the following formulation of Taylor's Inequality is sometimes useful.

Alternate Statement of Taylor's Inequality

If $\left|f^{(n+1)}(x)\right| \leq M$ for all x between a and x_0, then

$$\left|R_n(x_0)\right| \leq \frac{M}{(n+1)!}\left|x_0 - a\right|^{n+1}.$$

2. How large should n be to guarantee that the approximation of $e^{-0.1}$ using an nth degree Taylor polynomial for e^x centered at $a = 0$ is within 0.0001?

3. (a) Find the 2nd degree Taylor polynomial, $T_2(x)$ for $f(x) = x^{-2}$ based at $a = 1$.

   (b) Use Taylor's Inequality to estimate the accuracy of the approximation using $T_2(x)$ to estimate $f(x)$ in the interval $[0.9, 1.1]$.

4. How many terms of the Maclaurin series for $f(x) = \cos(2x)$ are needed to guarantee that the approximation of $f(x)$ on the interval $\left[-\frac{1}{3}, \frac{1}{3}\right]$ is within 10^{-6}?

5. Estimate the range of values of x for which the approximation

$$\ln(x) \approx \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2$$

is accurate to within 0.01.