

I) Some important relationships involving the position and momentum operators $\hat{X}$ and $\hat{P}$

see Schumm's QM ch. 4

A) a) $\langle \underline{r} |$ is a bra representing vector (x, y, z) position of a single ptcl.

b) $\langle \underline{p} |$ is a bra " " "Momentum, (p_x, p_y, p_z) , of a single ptcl.

c) The operators $\hat{X}$ and $\hat{P}$ are (implicitly) defined by the following:

$$\begin{aligned} \text{i) } \langle \underline{r} | \hat{X} | \psi \rangle &= x \langle \underline{r} | \psi \rangle \\ &= x \psi(\underline{r}) \end{aligned}$$

$$\text{ii) } \langle \underline{p} | \hat{P}_x | \psi \rangle = p_x \langle \underline{p} | \psi \rangle$$

iii) sim. relns hold for $\hat{y}, \hat{z}, \hat{p}_y, \hat{p}_z$

Thus, $\hat{X}$ operates on $\langle r |$:

yield $\langle r | \hat{X} = x \langle r |$

AND

$$\langle p | \hat{P}_x = p_x \langle p |$$

iv) ON THE NEXT PAGE, WE PROVE
that $|r\rangle$ is an eigenvector
of $\hat{X}$ (and of $\hat{Y}$ and of $\hat{Z}$)
with corresponding eigenvalue, x
(and y and z):

$$\hat{X}|r\rangle = x|r\rangle$$

$$\hat{Y}|r\rangle = y|r\rangle$$

$$\hat{Z}|r\rangle = z|r\rangle$$

v) similar relationships hold for $\hat{P}_x$ etc:

$$\hat{P}_x|p\rangle = p_x|p\rangle$$

$$\hat{P}_y|p\rangle = p_y|p\rangle$$

$$\hat{P}_z|p\rangle = p_z|p\rangle$$

(H.S.C) 120 11415 following pg. 51 (3) P.L
 B) Proof that $|\underline{r}\rangle$ is an eigenvector of the operator $\hat{X}$ with eigenvalue x .
 $\therefore \hat{X}|\underline{r}\rangle = x|\underline{r}\rangle$

$$\langle \underline{r}' | \hat{X} | \underline{r} \rangle = x \langle \underline{r}' | \underline{r} \rangle$$

* See solutions pg. 4.25

$$\begin{aligned}
 a) \quad \langle \underline{r}' | \hat{X} | \underline{r} \rangle &= x' \langle \underline{r}' | \underline{r} \rangle && (\text{since } \langle \underline{r}' | \hat{X} | \psi \rangle = x \langle \underline{r}' | \psi \rangle) \\
 &= x' \delta(\underline{r}' - \underline{r}) && (\text{since } \langle \underline{r}' | \underline{r} \rangle = \delta(\underline{r}' - \underline{r}))
 \end{aligned}$$

$$b) \text{ But } \langle \underline{r}' | \hat{X} | \underline{r} \rangle = \langle \underline{r} | \hat{X} | \underline{r}' \rangle$$

$\oplus$ Since $\hat{X}$ is Hermitian and since $|\underline{r}\rangle$ and $|\underline{r}'\rangle$ are real then

$$\hat{X}_{\underline{r}'\underline{r}} = \hat{X}_{\underline{r}\underline{r}'}^*$$

$$\text{or } \langle \underline{r}' | \hat{X} | \underline{r} \rangle = \langle \underline{r} | \hat{X} | \underline{r}' \rangle$$

$$c) \text{ So, } \langle \underline{r}' | \hat{X} | \underline{r} \rangle = \langle \underline{r} | \hat{X} | \underline{r}' \rangle = x \delta(\underline{r} - \underline{r}')$$

$$= x \langle \underline{r} | \underline{r}' \rangle \quad (\text{since } \langle \underline{r} | \underline{r}' \rangle = \delta(\underline{r} - \underline{r}'))$$

$$= x \langle \underline{r}' | \underline{r} \rangle \quad (\text{since } |\underline{r}\rangle \text{ and } |\underline{r}'\rangle \text{ real})$$

$$\langle \underline{r} | \underline{r}' \rangle =$$

$$\langle \underline{r}' | \underline{r} \rangle^* = \langle \underline{r}' |$$

d) THUS $\langle \underline{r}' | \hat{x} | \underline{r} \rangle = x \langle \underline{r}' | \underline{r} \rangle$ (Eq. 1)

e) since (Eq. 1) holds for any $\langle \underline{r}' |$,

then

$$\boxed{\hat{x} | \underline{r} \rangle = x | \underline{r} \rangle} \quad * \quad (\text{Eq. 2})$$

Similar arguments lead to, e.g.,

$$\hat{p}_x | \underline{p} \rangle = p_x | \underline{p} \rangle \text{ etc.}$$

(END ASIDE)

CONT'D

Ad) $\langle \underline{r}' | \underline{r} \rangle = \delta(\underline{r}' - \underline{r})$

e) $\langle \underline{p}' | \underline{p} \rangle = \delta(\underline{p}' - \underline{p})$

f) $\int | \underline{r}' \rangle \langle \underline{r} | d\underline{r} = \hat{I} = \delta_{\underline{r}' \underline{r}}$

$\int | \underline{r} \rangle \langle \underline{r} | d\underline{r} = \hat{I} = \delta_{\underline{r} \underline{r}}$

g) $\int | \underline{p}' \rangle \langle \underline{p} | d\underline{p} = \hat{I} = \delta_{\underline{p}' \underline{p}}$

h) $\boxed{\langle \underline{r} | \underline{p} \rangle = \langle \underline{p} | \underline{r} \rangle^* = \frac{1}{(2\pi\hbar)^{3/2}} e^{i \underline{p} \cdot \underline{r} / \hbar}}$

* Apply
to ALL FOR
to single
partic.

* these are
useful
identities
for doing
algebra/
calculus
during QM
calcs.

* useful for going from
 $\underline{r}$ to $\underline{p}$ basis or vice-versa

[lec. 18:5]

$$i) \quad \langle \underline{r} | \psi \rangle = \psi(\underline{r}) = \int \langle \underline{r} | \underline{p} \rangle \langle \underline{p} | \psi \rangle d\underline{p} \quad (\text{p.t. 1})$$

$$\langle \underline{r} | \hat{I} | \psi \rangle \xrightarrow{\quad} \boxed{\psi(\underline{r}) = \int \frac{1}{(2\pi\hbar)^{3/2}} e^{\frac{i}{\hbar} \underline{p} \cdot \underline{r}} \tilde{\psi}(\underline{p}) d\underline{p}} \quad \left(\begin{array}{l} \text{simple} \\ \text{p.t. 1} \end{array} \right) \quad (5.1)$$

$\therefore \psi(\underline{r})$ is Fourier Transform of $\tilde{\psi}(\underline{p})$

$$j) \quad \langle \underline{p} | \psi \rangle = \tilde{\psi}(\underline{p}) = \int \langle \underline{p} | \underline{r} \rangle \langle \underline{r} | \psi \rangle d\underline{r}$$

$$\langle \underline{p} | \hat{I} | \psi \rangle \xrightarrow{\quad} \langle \underline{r} | \underline{p} \rangle^* \xrightarrow{\quad} \boxed{\tilde{\psi}(\underline{p}) = \int \frac{1}{(2\pi\hbar)^{3/2}} e^{-\frac{i}{\hbar} (\underline{p} \cdot \underline{r})} \psi(\underline{r}) d\underline{r}} \quad \left(\begin{array}{l} \text{simple} \\ \text{p.t. 2} \end{array} \right) \quad (5.2)$$

$\therefore \tilde{\psi}(\underline{p})$ and $\psi(\underline{r})$ are F.T. Transform pairs

k) CANONICAL
COMUTATION RELATIONS

$$[\hat{R}_i, \hat{P}_j] = i\hbar \quad \text{for } (\hat{R}_1 = \hat{x}, \hat{R}_2 = \hat{y}, \hat{R}_3 = \hat{z})$$

$$[\hat{R}_i, \hat{R}_j] = 0$$

$$[\hat{P}_i, \hat{P}_j] = 0$$

(5.3)

B) Example/practice prob. 2

(see Schaum's QM prob. 4.26)

prove that

$$\begin{aligned}\langle \underline{r} | \hat{p} | \psi \rangle &= -i\hbar \nabla \langle \underline{r} | \psi \rangle \\ &= -i\hbar \nabla \psi(\underline{r})\end{aligned}$$

a) Try inserting $\hat{I}$:

$$\begin{aligned}\langle \underline{r} | \hat{p} | \psi \rangle &= \langle \underline{r} | \hat{I} \hat{p} | \psi \rangle \quad \underline{p} \tilde{\psi}(\underline{p}) \\ &= \int \langle \underline{r} | \underline{p} \rangle \langle \underline{p} | \hat{p} | \psi \rangle d\underline{p} \quad \underline{p} \tilde{\psi}(\underline{p}) \\ &= \int \frac{1}{(2\pi\hbar)^3} \exp[i\underline{p} \cdot \underline{r}] \underline{p} \tilde{\psi}(\underline{p}) d\underline{p} \\ &= \text{F.T. of } \underline{p} \tilde{\psi}(\underline{p}) \\ &= -i\hbar \nabla \psi(\underline{r}) \quad \downarrow \begin{pmatrix} \text{I did} \\ \text{not prove it} \\ \text{this} \end{pmatrix}\end{aligned}$$

c) Practice prob. 3 = Schaum's QM
prob. 4.27

prove $[\hat{x}, \hat{p}_x]$

$$a) X[\hat{x}, \hat{p}_x]|\psi\rangle = \langle \hat{x} \hat{p}_x |\psi\rangle - \langle \hat{p}_x \hat{x} |\psi\rangle$$

$$b) x \langle r | \hat{p}_x |\psi\rangle + i\hbar \frac{\partial}{\partial x} \langle r | x |\psi\rangle$$

$$c) = x \frac{\hbar}{i} \frac{\partial}{\partial x} \langle r | \psi \rangle - \frac{\hbar}{i} \frac{\partial}{\partial x} (x \langle r | \psi \rangle)$$

$$= x \frac{\hbar}{i} \frac{\partial}{\partial x} \psi(r) - \frac{\hbar}{i} x \frac{\partial}{\partial x} \psi(r) - \frac{\hbar}{i} \psi(r)$$

$$\therefore d) \langle r | [\hat{x}, \hat{p}_x] |\psi\rangle = i\hbar \langle r | \psi \rangle (= \langle r | \hat{I} i\hbar |\psi\rangle)$$

$$\therefore e) \boxed{[\hat{x}, \hat{p}_x] = i\hbar}$$

c) A THIRD PRACTICE PROBLEM
is given @ the end of the
lec. 17 NOTES.

Postulation 1 Rough notes
Lec. 16
Schrodinger p. 9 ①

I) ^{dyn} state $\Rightarrow |\psi(t)\rangle$ $|\psi(0)\rangle$ v/r

II) $H(x, p)$ in classical model

$$\Rightarrow \hat{H}(\hat{x}, \hat{p}) \quad \text{where } \hat{x} = \hat{p} = \hat{p}$$

$$\hat{x} = x |x\rangle\langle x|$$

$$\Rightarrow \langle x' | \hat{x} | x \rangle = x \langle x' | x \rangle$$

$$\hat{x} | x \rangle = x | x \rangle$$

$$\langle x' | \hat{x} | x \rangle = x \langle x' | x \rangle = x \delta(x - x')$$

Schrodinger

rep'd by vector

I) state $\in |\psi\rangle$ is a Hilbert space, S

II)

Any measurable observable, $\hat{O}$,

$\Rightarrow$ operator $\hat{O}$ w/in S

a) $\hat{O}$ is a matrix on

b) $\hat{O}$ is hermitian

$$\Rightarrow \hat{O}_{ab} = \hat{O}_{ba}^*$$

c) eigenvalues of $\hat{O}$ are real

d) eigenvectors of $\hat{O}$ are orthonormal

a complete orthogonal set, $\{ |w_i\rangle \}$
 e) from d) $|\psi\rangle$ can be stated in
 the basis of $\mathcal{R}$

$$\Rightarrow |\psi\rangle = \sum_i |w_i\rangle \langle w_i | \psi \rangle \quad (2.1)$$

now, let $\hat{I} = \sum_i |w_i\rangle \langle w_i|$

e) for an operator $\hat{A}$ having continuous
 spectrum $\mathcal{L}(\omega)$,

$$|\psi\rangle = \int |w\rangle \langle w | \psi \rangle d\omega \quad (2.2)$$

now (2.2) $\Rightarrow$

$$\hat{I} = \int |w\rangle \langle w| d\omega$$

f) the prob. of observing any
 given eigenvalue associated w/
 $\mathcal{A}$

$$P(w_i) = \langle w_i | \psi \rangle^2$$

$$= \langle w_i | \psi \rangle \langle \psi | w_i \rangle$$

$$= \langle \psi | w_i \rangle \langle w_i | \psi \rangle \text{ discrete}$$

(3)

$$P(\omega_i) = \langle \psi | \hat{P}_{\omega_i} | \psi \rangle$$

$$P_{\omega_i} = |\omega_i\rangle\langle\omega_i|$$

for non-degen.
 ω_i

for degen. ω_i of degree m

$$\begin{aligned} P(\omega_i) &= \sum_{j=1}^m \langle \psi | \omega_{ij} \rangle \langle \omega_{ij} | \psi \rangle = \sum_{j=1}^m \langle \omega_{ij} | \psi \rangle \\ &= \langle \psi | \Sigma | \psi \rangle \\ &= \langle \psi | \hat{P}_\omega | \psi \rangle \end{aligned}$$

==

CONTINUOUS SPECTRUM: ω

$$\begin{aligned} \text{Prob } P(\omega) &= |\langle \omega | \psi \rangle|^2 \\ &= \langle \psi | \omega \rangle \langle \omega | \psi \rangle \\ &= \rho d\omega \end{aligned}$$

==

prob. of state of system after
a measurement

(4)

~~Assume we have a system~~
ASIDE

g) tests of QM.

a) ~~setup~~ prepare an ensemble
of ptcls ~~at~~ or quanta
systems in a ~~of~~ known
state, $|\psi\rangle$

ex. 1 photons of known
polarization in a laser
beam

ex. 2 He^{2+} ptcl beam

~~ex. 3~~

ii) measure an obs. e.g.
position or momentum
and ~~measure~~ ^{measure} ~~difference~~, e.g.,

$\langle x \rangle_{\text{exp}}$ and/or $\langle p_x \rangle_{\text{exp}}$ $(\Delta x), (\Delta p_x)$

iii) compare exp w/ theory

(5)

~~Q10)~~

$$\begin{aligned}
 \langle \hat{r} \rangle &\equiv \sum_i \psi^\dagger w_i P(w_i) \psi \\
 &= \sum_i w_i \langle \psi | \hat{r} | \psi \rangle P(w_i) \\
 &\quad \text{(non-deg.)}
 \end{aligned}$$

But $\hat{r} |w_i\rangle = w_i |w_i\rangle$

$$\begin{aligned}
 \langle \hat{r} \rangle &= \sum_i \psi^\dagger \hat{r} |w_i\rangle \langle w_i | \psi \rangle \\
 &= \langle \psi | \hat{r} | \psi \rangle
 \end{aligned}$$

$$\begin{aligned}
 \langle \Delta \hat{r}^2 \rangle &= \sum_i \psi^\dagger P(w_i) (w_i - \langle \omega \rangle)^2 \psi \\
 &= \sum_i P(w_i) [w_i^2 - \langle \omega \rangle^2] \\
 &= \sum_i \langle \psi | w_i \rangle \langle w_i | \psi \rangle [\quad]
 \end{aligned}$$

~~But $\hat{r} |w_i\rangle = w_i |w_i\rangle$~~

$$= \langle \psi | (\hat{r} - \langle \hat{r} \rangle)^2 | \psi \rangle$$

to be proved

(+) check $\langle \hat{r} \rangle_{\text{exp}}$ vs $\langle \hat{r} \rangle_{\text{theor}}$
 " $\langle \Delta \hat{r} \rangle_{\text{exp}}$ vs $\langle \Delta \hat{r} \rangle_{\text{theor}}$

6

n) Collapse of $|\psi\rangle$ during measurement of $\hat{Q}$

II) QUANTUM MEASUREMENTS

A) The only possible outcomes of ~~an~~ measurement of obs. $\hat{Q}$ are the eigenvalues of $\hat{Q}$.

B) Prob of obs. w_i :

$$P(w_i) = |\langle w_i | \psi \rangle|^2$$

C) If given an observed w_i :
the ~~initial~~ pre-measurement state $|\psi\rangle$ collapses to a known state $|w_i\rangle$

$\hat{Q} =$ the eigenstate (non-deg. w_i)

$=$
or a known eigenspace spanned by the eigenvectors ass'd w/ w_i (degenerate)

III) evolution of $|\psi\rangle$ goes
By Schrodinger eqn.

(7)

Q. 11.

⊕ $H \Rightarrow \hat{H}$ involves modeling

⊕ prob/validity of
model comes via
experiments or
numerical sim.

⊕ ex. V =

AS108: Review on finding eigenvalues
& eigenvectors of a given
matrix A

find a 's from characteristic eqn

$$= \det \hat{A}|a\rangle = a|a\rangle$$

$$\Rightarrow (\hat{A} - \hat{I}a)|a\rangle = 0$$

$$\Rightarrow \det(\hat{A} - \hat{I}a) = 0$$

→ char. eqn for a 's
for $N \times N \hat{A} \Rightarrow a$ roots

it ~~with~~ a root reports $\Rightarrow$
deg. eigenvalue.

⑧ ~~or~~ given $a_1, a_2, \dots, a_N$
to get $\lambda(a_i)$
~~is~~

$$\hat{A}(a_i) = a_i(a_i)$$

↳ N eqns in N unk's

$$a_1, a_2, \dots, a_N$$