

Sum 24 Exam 5 Practice Problems - Solutions

$$\textcircled{1} \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

$$\text{AST: } b_n = \frac{1}{\sqrt{n+1}}$$

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1}} = 0 \checkmark$$

$$b_{n+1} = \frac{1}{\sqrt{n+2}} \leq \frac{1}{\sqrt{n+1}} = b_n \quad \text{so } b_n \text{ is decreasing} \checkmark$$

The series converges by AST.

$$\textcircled{2} \sum_{n=1}^{\infty} \frac{n^2}{(n+1)!}$$

$$\text{Ratio Test: } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{((n+1)+1)!} \cdot \frac{(n+1)!}{n^2} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2 \cdot (n+1)!}{(n+2)! \cdot n^2} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2 \cdot (n+1)!}{(n+2) \cdot (n+1)! \cdot n^2} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{(n+2)n^2} \right|$$

$$= 0 \quad \text{because degree of numerator is 2 and degree of denominator is 3 (larger so limit 0)}$$

Since $L=0 < 1$, the series converges by the Ratio Test

$$(3) (a) \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5+1}}$$

LCT: $a_n = \frac{n}{\sqrt{n^5+1}} \geq 0$ ✓

Pick $b_n = \frac{n}{\sqrt{n^5}} = \frac{n}{n^{5/2}} = \frac{1}{n^{3/2}} \geq 0$ ✓

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a p-series with $p = \frac{3}{2} > 1$ so converges.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\frac{n}{\sqrt{n^5+1}}}{\frac{1}{n^{3/2}}} \\ &= \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^5+1}} \cdot \frac{n^{3/2}}{1} \\ &= \lim_{n \rightarrow \infty} \frac{n^{5/2}}{\sqrt{n^5+1}} \quad \begin{array}{l} \nwarrow \text{degrees match,} \\ \swarrow \text{both } 5/2 \end{array} \\ &= 1 \end{aligned}$$

Since $0 < 1 < \infty$, both series converge.

$$(b) \sum_{n=1}^{\infty} \frac{1+\sin^2(n)}{n^3}$$

DCT: $a_n = \frac{1+\sin^2(n)}{n^3} \geq 0$ ✓

Pick $b_n = \frac{1+1}{n^3} = \frac{2}{n^3} \geq 0$ ✓

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2}{n^3}$ converges, $p=3 > 1$

Also $\frac{1+\sin^2(n)}{n^3} \leq \frac{2}{n^3}$ because $\sin^2(n) \leq 1$

Therefore, $\sum_{n=1}^{\infty} a_n$ converges by Direct Comparison Test.

$$\textcircled{4} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{4/3}}$$

Notice that $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^{4/3}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{4/3}} \leftarrow p = 4/3 > 1$
so converges.

Therefore, the series converges absolutely.

$$\textcircled{5} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

(AST Remainder Estimate: $|R_n| \leq b_{n+1}$)

For this series, $b_n = \frac{1}{\sqrt{n}}$ so the remainder estimate says

$$|R_n| \leq b_{n+1} = \frac{1}{\sqrt{n+1}} \leq 0.0001$$

error $\nearrow$

$$\frac{1}{\sqrt{n+1}} \leq \frac{1}{10000}$$

$$10000 \leq \sqrt{n+1}$$

$$(10000)^2 \leq n+1$$

$$(10000)^2 - 1 \leq n$$

$$\text{Answer: } n = (10000)^2 - 1$$

$$(6) \sum_{n=0}^{\infty} e^{-2n}$$

(a) Int Test: (don't need to check conditions)

$$\begin{aligned} \int_0^{\infty} e^{-2x} dx &= \lim_{t \rightarrow \infty} \int_0^t e^{-2x} dx \\ &= \lim_{t \rightarrow \infty} \left(-\frac{1}{2} e^{-2x} \right) \Big|_0^t \\ &= \lim_{t \rightarrow \infty} \left(-\frac{1}{2} e^{-2t} + \frac{1}{2} e^0 \right) \\ &= 0 + \frac{1}{2} \end{aligned}$$

Since the improper integral converges, the series converges by the Integral Test.

(b) Int. Test. Remainder Estimate says

$$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx \quad \left(\begin{array}{l} \text{same } f \text{ as} \\ \text{in Int. Test} \end{array} \right)$$

$$\begin{aligned} R_n &\leq \int_n^{\infty} e^{-2x} dx = \lim_{t \rightarrow \infty} \int_n^t e^{-2x} dx \\ &= \lim_{t \rightarrow \infty} \left(-\frac{1}{2} e^{-2x} \right) \Big|_n^t \\ &= \lim_{t \rightarrow \infty} \left(-\frac{1}{2} e^{-2t} + \frac{1}{2} e^{-2n} \right) \\ &= 0 + \frac{1}{2} e^{-2n} \end{aligned}$$

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$$R_n \leq \frac{1}{2} e^{-2n} \leq 0.005$$

$$\frac{1}{2} e^{-2n} \leq \frac{5}{1000}$$

$$e^{-2n} \leq \frac{10}{1000}$$

$$\frac{1}{e^{2n}} \leq \frac{1}{100}$$

$$100 \leq e^{2n}$$

$$\ln(100) \leq 2n$$

$$\frac{\ln(100)}{2} \leq n$$

$$\textcircled{7} \quad f(x) = \sqrt{x}, \quad a=1$$

$$f(x) = x^{1/2}$$

$$f(1) = 1$$

$$C_0 = 1$$

$$f'(x) = \frac{1}{2} x^{-1/2}$$

$$f'(1) = \frac{1}{2}$$

$$C_1 = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4} x^{-3/2}$$

$$f''(1) = -\frac{1}{4}$$

$$C_2 = \frac{(-1/4)}{2!} = -\frac{1}{8}$$

$$f'''(x) = \frac{3}{8} x^{-5/2}$$

$$f'''(1) = \frac{3}{8}$$

$$C_3 = \frac{(3/8)}{3!} = \frac{1}{16}$$

$$T_3(x) = C_0 + C_1(x-1) + C_2(x-1)^2 + C_3(x-1)^3$$

$$= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$$

⑧ $\sum_{n=2}^{\infty} \frac{(x-5)^n}{2^n \ln n}$ ← base point is $a=5$

↑ series should start at 2 to avoid dividing by zero

Ratio Test: $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-5)^{n+1}}{2^{n+1} \ln(n+1)} \cdot \frac{2^n \ln n}{(x-5)^n} \right|$

$$= \lim_{n \rightarrow \infty} \left| \frac{(x-5)^n (x-5) \cdot 2^n \cdot \ln n}{2^n \cdot 2 \cdot \ln(n+1) \cdot (x-5)^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(x-5) \ln n}{2 \ln(n+1)} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{|x-5| \ln n}{2 \ln(n+1)}$$

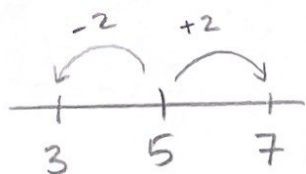
$$= \frac{|x-5|}{2} \lim_{n \rightarrow \infty} \frac{\ln n}{\ln(n+1)}$$

← $\frac{\infty}{\infty}$
L'Hospital's Rule

$$\stackrel{LH}{=} \frac{|x-5|}{2} \lim_{n \rightarrow \infty} \underbrace{\frac{1/n}{1/(n+1)}}_{\text{goes to 1}} = \frac{|x-5|}{2}$$

The series converges if $L = \frac{|x-5|}{2} < 1$ (by Ratio Test)

$$|x-5| < 2 \quad \text{so } \boxed{R=2}$$



End points are $x=3$ and $x=7$. Need to check endpoints to get the IOC.

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$$\text{At } x=7: \sum_{n=2}^{\infty} \frac{(7-5)^n}{2^n \ln n} = \sum_{n=2}^{\infty} \frac{2^n}{2^n \ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n} \quad \leftarrow \text{compare to } \sum \frac{1}{n}$$

Comparison Test: $a_n = \frac{1}{\ln n} \geq 0$ ✓

Pick $b_n = \frac{1}{n} \geq 0$ ✓

$$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n} \text{ diverges (harmonic)}$$

Want: $b_n \leq a_n$

$$\frac{1}{n} \leq \frac{1}{\ln n} \quad \leftarrow \text{true because } \ln(n) \leq n \text{ for all } n$$

So $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by the Comparison Test.
 $\rightarrow x=7$ is not in IOC

$$\text{At } x=3: \sum_{n=2}^{\infty} \frac{(3-5)^n}{2^n \ln n} = \sum_{n=2}^{\infty} \frac{(-2)^n}{2^n \ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n 2^n}{2^n \ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$$

AST: $b_n = \frac{1}{\ln n}$

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0 \quad \checkmark$$

b_n is decreasing ✓ because $\ln n$ is increasing
 (1 over an increasing seq. is decreasing)

So $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is convergent by AST.

$\rightarrow x=3$ is included in IOC

The interval of convergence is $[3, 7)$.