

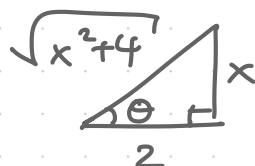
Trig Sub continued

example: $\int_0^2 \frac{x^3}{\sqrt{x^2+4}} dx$



$$x = 2 \tan \theta$$

$$dx = 2 \sec^2 \theta d\theta$$



from triangle,

$$\frac{\sqrt{x^2+4}}{2} = \sec \theta$$

$$\sqrt{x^2+4} = 2 \sec \theta$$

change bounds: $x=0 \Rightarrow \tan \theta = 0$

$$\theta = 0$$

$$x=2 \Rightarrow \tan \theta = 1$$

$$\theta = \pi/4$$

$$\begin{aligned} \int_0^2 \frac{x^3}{\sqrt{x^2+4}} dx &= \int_0^{\pi/4} \frac{(2 \tan \theta)^3 \cdot 2 \sec^2 \theta}{2 \sec \theta} d\theta \\ &= 8 \int_0^{\pi/4} \tan^3 \theta \sec \theta d\theta \end{aligned}$$

$$\begin{aligned} u &= \sec \theta \\ du &= \sec \theta \tan \theta d\theta \end{aligned}$$

OR

~~$$\begin{aligned} u &= \tan \theta \\ du &= \sec^2 \theta d\theta \end{aligned}$$~~

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$= 8 \int_0^{\pi/4} \underbrace{\tan^2 \theta}_{\text{convert to sec } \theta} \cdot \underbrace{\tan \theta \sec \theta d\theta}_{du}$$

$$= 8 \int_0^{\pi/4} (\sec^2 \theta - 1) \cdot \tan \theta \sec \theta d\theta$$

$$\boxed{\begin{array}{l} u = \sec \theta \\ du = \sec \theta \tan \theta d\theta \end{array}}$$

$$\begin{aligned} \theta = 0 &\Rightarrow u = \sec(0) \\ &= \frac{1}{\cos(0)} = 1 \end{aligned}$$

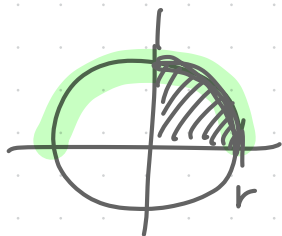
$$\theta = \pi/4 \Rightarrow u = \sec(\pi/4) = \sqrt{2}$$

$$= 8 \int_1^{\sqrt{2}} u^2 - 1 du$$

$$= 8 \left(\frac{1}{3} u^3 - u \right) \Big|_1^{\sqrt{2}}$$

$$= 8 \left(\left(\frac{1}{3} (\sqrt{2})^3 - \sqrt{2} \right) - \left(\frac{1}{3} (1)^3 - 1 \right) \right)$$

example: Show that the area of a circle of radius r is πr^2 .
(written HW #2)



Recall $y = \sqrt{r^2 - x^2}$ is the top half of a circle of radius r .

We'll find the shaded area and multiply by 4 at the end.

$$\int_0^r \sqrt{r^2 - x^2} dx$$



$$\begin{aligned} x &= r \sin \theta \\ dx &= r \cos \theta d\theta \end{aligned}$$

change bounds

$$\begin{aligned} x=0 &\Rightarrow r \sin \theta = 0 \\ \theta &= 0 \end{aligned}$$

$$\begin{aligned} x=r &\Rightarrow r \sin \theta = r \\ \sin \theta &= 1, \theta = \frac{\pi}{2} \end{aligned}$$

from triangle,

$$\frac{\sqrt{r^2 - x^2}}{r} = \cos \theta$$

$$\sqrt{r^2 - x^2} = r \cos \theta$$

$$\int_0^r \sqrt{r^2 - x^2} dx = \int_0^{\pi/2} r \cos \theta \cdot r \cos \theta d\theta$$

$$= r^2 \int_0^{\pi/2} \cos^2 \theta d\theta$$

$$= r^2 \int_0^{\pi/2} \frac{1}{2} (1 + \cos(2\theta)) d\theta$$

$$= \frac{r^2}{2} \int_0^{\pi/2} 1 + \cos(2\theta) d\theta$$

$$= \frac{r^2}{2} \left[\theta + \frac{1}{2} \sin(2\theta) \right]_0^{\pi/2}$$

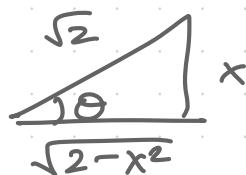
$$= \frac{r^2}{2} \left(\frac{\pi}{2} + \frac{1}{2} \sin(\pi) - 0 \right)$$

$$= \frac{\pi r^2}{4}$$

So area of circle is $4 \left(\frac{\pi r^2}{4} \right) = \pi r^2$

example: $\int \frac{x^2}{\sqrt{2-x^2}} dx$

$a = \sqrt{2}$



$x = \sqrt{2} \sin \theta$

$dx = \sqrt{2} \cos \theta d\theta$

from triangle,

$\frac{\sqrt{2-x^2}}{\sqrt{2}} = \cos \theta$

$\sqrt{2-x^2} = \sqrt{2} \cos \theta$

$\rightarrow = \int \frac{(\sqrt{2} \sin \theta)^2}{\cancel{\sqrt{2}} \cancel{\cos \theta}} \cdot \cancel{\sqrt{2}} \cancel{\cos \theta} d\theta$

$= \int 2 \sin^2 \theta d\theta$

$= 2 \int \frac{1}{2} (1 - \cos(2\theta)) d\theta$

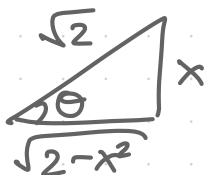
$= \int 1 - \cos(2\theta) d\theta$

$= \theta - \frac{1}{2} \sin(2\theta) + C$



Trick: use $\sin(2\theta) = 2\sin\theta\cos\theta$

so we can use our triangle to
sub back



$$\begin{aligned} 2\sin\theta\cos\theta &= \cancel{2} \left(\frac{x}{\cancel{\sqrt{2}}} \right) \left(\frac{\sqrt{2-x^2}}{\cancel{\sqrt{2}}} \right) \\ &= x\sqrt{2-x^2} \end{aligned}$$

For θ : $x = \sqrt{2}\sin\theta$

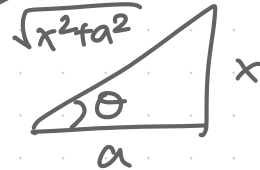
$$\frac{x}{\sqrt{2}} = \sin\theta \Rightarrow \underline{\theta = \arcsin\left(\frac{x}{\sqrt{2}}\right)}$$

Answer: $\theta - \frac{1}{2}\sin(2\theta) + C$

$$= \arcsin\left(\frac{x}{\sqrt{2}}\right) - \frac{1}{2}x\sqrt{2-x^2} + C$$

example: $\int \frac{1}{x^2+a^2} dx$, a is a constant

$$\int \frac{1}{x^2+a^2} dx = \int \frac{1}{(\sqrt{x^2+a^2})^2} dx$$



$$x = a \tan \theta$$
$$dx = a \sec^2 \theta d\theta$$

$$\frac{\sqrt{x^2+a^2}}{a} = \sec \theta$$

$$\sqrt{x^2+a^2} = a \sec \theta$$

$$\rightarrow = \int \frac{a \sec^2 \theta}{(a \sec \theta)^2} d\theta$$

$$\tan \theta = \frac{x}{a}$$

$$\theta = \arctan\left(\frac{x}{a}\right)$$

$$= \int \frac{\cancel{a} \sec^2 \theta}{a^2 \cancel{\sec^2 \theta}} d\theta$$

$$= \int \frac{1}{a} d\theta = \frac{1}{a} \int 1 d\theta$$

$$= \frac{1}{a} \theta + C$$

$$= \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

Memorize

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

Notice if $a=1$: $\int \frac{1}{x^2+1} dx = \arctan(x) + C$

$$\int \frac{1}{x^2+4} dx = \frac{1}{2} \arctan\left(\frac{x}{2}\right) + C$$

BUT $\int \frac{1}{x^2-4} dx$ is NOT an
arctan integral!

$$\int x \sec^2 x dx$$

$\swarrow$ power $\nwarrow$ trig

IBP - LIPE T

$\swarrow$ power of x
 $\nwarrow$ trig fun

$$u = x$$

$$du = dx$$

$$dv = \sec^2 x dx$$

$$v = \tan x$$

$$\int \sin^2 x \cos^5 x dx$$

$$\int \sin^2 x \cdot \underbrace{\cos^4 x}_{\substack{\text{convert} \\ \text{to sinx}}} \cdot \underbrace{\cos x dx}_{du}$$

$$\sin^2 x + \cos^2 x = 1$$

$$\cos^2 x = 1 - \sin^2 x$$

$$u = \sin x$$

$$du = \cos x dx \quad \leftarrow$$

$$= \int \sin^2 x (\cos^2 x)^2 \cdot \cos x dx$$

$$= \int \sin^2 x (1 - \sin^2 x)^2 \underbrace{\cos x dx}$$

$$= \int u^2 (1 - u^2)^2 du$$

$$\int \sin^5 x \cos^2 x \, dx$$

$$= \int \underbrace{\sin^4 x \cos^2 x}_{\text{convert to cos x}} \cdot \underbrace{\sin x \, dx}_{(-1) \, du}$$

$$u = \cos x$$

$$du = -\sin x \, dx$$