ISSN 0005-1179, Automation and Remote Control, 2021, Vol. 82, No. 7, pp. 1248–1261. © Pleiades Publishing, Ltd., 2021.
Russian Text © The Author(s), 2020, published in Upravlenie Bol'shimi Sistemami, 2020, No. 86, pp. 32–54.

LARGE SCALE SYSTEMS CONTROL

Robust Stabilization of Linear Plants in the Presence of Disturbances and High-Frequency Measurement Noise

I. B. Furtat[1*], A. N. Nekhoroshikh[1], and P. A. Gushchin[1***]**

[1] *Institute for Problems in Mechanical Engineering, Russian Academy of Sciences, St. Petersburg, 199178 Russia*

*e-mail: *cainenash@mail.ru, **becks94@mail.ru, ***guschin.p@mail.ru*

Received April 24, 2020; revised April 24, 2020; accepted July 31, 2020

Abstract—We propose a solution of the robust stabilization problem for linear dynamic plants with unknown parameters belonging to a known compact set, bounded exogenous disturbances, and bounded high-frequency measurement noise. The control algorithm synthesis is divided into two stages. The filtering algorithm synthesized at the first stage permits one to reduce the influence of the measurement noise on the plant output variable. Constructive conditions for selecting the filtering algorithm parameters are proposed for the case in which the measurement noise can be represented as a sum of sinusoidal signals. At the second stage, we synthesize a control algorithm suppressing the influence of the parametric uncertainty and exogenous disturbances. This algorithm is based on the use of finite differences in continuous time; this allows avoiding the use of dynamic observers increasing the dimension of the closed-loop system. Simulation results illustrating the efficiency of our algorithm in comparison with some existing analogs are presented. A comparative analysis with the results by Astolfi et al. has shown that our control algorithm has lower dynamic order and guarantees higher accuracy in the output signal and its derivatives. Moreover, the algorithm parameter selection in our algorithm is easier owing to the independent adjustment of the filter and control law in contrast to the results by Astolfi et al., where the controller parameters are selected simultaneously for the entire algorithm.

Keywords: linear plant, high-frequency measurement noise, filter, disturbance, control

DOI: 10.1134/S0005117921070080

1. INTRODUCTION

The synthesis of algorithms that ensure reaching a given goal under conditions of uncertainty in the parameters of the mathematical model of the plant and the presence of exogenous disturbances is a topical problem in automatic control theory. Currently, many various approaches have been proposed to solving this problem, for example, H-optimal control, adaptive control, robust control, nonlinear control, etc. (see, e.g., [4, 5]).

One common method for solving the control problem under uncertainty is to synthesize controllers with the use of estimates of the derivatives of the controlled variable based on a high-gain observer. A high-gain observer was first proposed in [11, 13] for minimum-phase plants. Later, many various modifications of high-gain observers were obtained for minimum-phase plants; see, e.g., [1, 9, 10, 17, 20]. It was shown in [1, 9, 10, 14, 17, 20] that, to construct a control system, it suffices to use an observer whose dimension is at least $\gamma - 1$, where γ is the plant relative degree.

However, the use of a high-gain observer may prove unsatisfactory in the presence of high-frequency measurement noise. More detailed studies can be found in [13, 19, 21]. The problem was that the estimate of the derivative of the controlled variable could significantly exceed its true

value. In this case, the error is accumulated in the subsequent estimates of the derivatives. This problem is associated with both the differentiation of the high-frequency noise and the presence of high gain in the control system.

To partially overcome this problem, it was proposed in [13, 18, 19] to adjust the high gain of the observer adaptively. Thus, to improve the convergence of the processes, the observer gain took large values at the initial moment of system operation and was then decreased in the steady state.

The papers [12, 22] propose a new observer, which is an extension of the high-gain observer in [14] to the case of high-frequency measurement noise. In this case, the dimension of the new observer is $2\gamma - 2$. The twofold increased dimension of the observer in [12, 22] in comparison with the observer in [14] is due to the introduction of additional differential equations that compensate for the influence of the high-frequency measurement noise. The papers [12, 22] give a comparative numerical analysis of that algorithm and the algorithm in [14], which shows the efficiency of using the algorithm in [12, 22] in comparison with the one in [14].

However, to calculate the parameters of the observer in [12, 22], it is necessary to solve an equation of the order $2\gamma - 2$. In this case, the quality of estimation of the derivatives and the quality of filtering properties simultaneously depend on the solutions of this equation. Moreover, the quality of the filtering properties of the observers [11, 12, 18, 19, 22] is substantially limited.

In the present paper, unlike [12, 22], we propose to separate the filtering and estimation properties in the form of two independent algorithms. As a result, compared to [11, 12, 18, 19, 22], this permits one

1. To improve the quality of filtering of the high-frequency noise.
2. To make the calculations of the filter and observer parameters independent.
3. To use any of the existing observers to estimate the derivatives.

The article deals with the robust stabilization problem for a plant with unknown parameters that is affected by bounded disturbances and whose output is measured with high-frequency noise. To synthesize a control system, we use an approach that permits independently controlling the quality of noise filtering and the quality of the output-variable stabilization error. Simulation results are presented, which illustrate the efficiency of our algorithm in comparison with the classical high-gain observer in [14] and the high-gain observer under measurement noise conditions in [12, 22].

2. STATEMENT OF THE PROBLEM

Consider a plant with a model described by the equations

$$Q(p)z(t) = R(p)u(t) + f(t), \quad y(t) = z(t) + w(t), \tag{1}$$

where $y(t) \in R$ is the controlled variable, $u(t) \in R$ is the control signal, $f(t) \in R$ is an uncontrolled exogenous bounded disturbance, $w(t) \in R$ is an exogenous high-frequency bounded noise, $Q(p) = p^n + q_{n-1}p^{n-1} + \cdots + q_0$ and $R(p) = r_m p^m + r_{m-1}p^{n-1} + \cdots + r_0$ are linear differential operators with unknown coefficients $q_{n-1}, \ldots, q_0$ and $r_m, \ldots, r_0$, respectively, and $p = d/dt$ is the differentiation operator.

The plant is subjected to the following assumptions when solving the problem.

Assumption 1. *The unknown coefficients $q_{n-1}, \ldots, q_0$ and $r_m, \ldots, r_0$ belong to a known compact set Ξ.*

Assumption 2. *The polynomial $R(\lambda)$ is Hurwitz, where λ is a complex variable and $r_m > 0$.*

Assumption 3. *Only the signal $y(t)$ is accessible for measurement in the plant.*

It is required to construct a continuous-time control system that ensures the validity of the target condition

$$\overline{\lim_{t\to\infty}}\big|z(t)\big| < \delta_1, \tag{2}$$

where $\delta_1 > 0$ is the control accuracy; here and in the following, $|\cdot|$ is the Euclidean norm of a vector.

3. HIGH-FREQUENCY MEASUREMENT NOISE FILTERING ALGORITHM

To extract the signal z from the signal y, we consider the algorithm

$$\mu\dot{\xi}(t) = G\xi(t) + By(t), \quad \hat{y}(t) = L_r\xi(t), \quad \xi(0) = 0, \tag{3}$$

where $\xi = [\xi_1, \xi_2, \ldots, \xi_r]^{\mathrm{T}}$,

$$G = \begin{bmatrix} -\sigma_1^{-1} & 0 & 0 & \ldots & 0 & 0 \\ \sigma_2^{-1} & -\sigma_2^{-1} & 0 & \ldots & 0 & 0 \\ 0 & \sigma_3^{-1} & -\sigma_3^{-1} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \sigma_r^{-1} & -\sigma_r^{-1} \end{bmatrix},$$

$\sigma_i > 0$, $B = \big[\sigma_1^{-1}, 0, \ldots, 0\big]^{\mathrm{T}}$, and $\mu > 0$ is a small parameter. Here and in the following, the matrix $L_l = [0, \ldots, 0, l]$ is of size $1 \times l$. For example, the size of L_r in (3) is $1 \times r$.

Assertion 1. *Let the above assumptions be true, let the signal z be bounded, and let the following relation hold for w:*

$$\mu^{-1}\overline{\lim_{t\to\infty}}\left|\int_0^t e^{\mu^{-1}G(t-s)}Bw(s)ds\right| < \delta_2, \tag{4}$$

where $\delta_2 > 0$ is a sufficiently small number. Then there exists a $\mu_0 > 0$ such that the condition

$$\overline{\lim_{t\to\infty}}\big|\hat{y}(t) - z(t)\big| < \delta_3 \tag{5}$$

is satisfied for $\mu \le \mu_0$; here $\delta_3 > 0$ is a sufficiently small number.

Proof. Consider system (3) with the signal z fed to the input instead of the signal y,

$$\mu\dot{\widetilde{\xi}}(t) = G\widetilde{\xi}(t) + Bz(t), \quad \widetilde{y}(t) = L_r\widetilde{\xi}(t), \quad \widetilde{\xi}(0) = 0. \tag{6}$$

To analyze system (6), we use Lemma 1 [8, 15, 16].

Lemma 1. *Consider the system*

$$\dot{x}(t) = f\big(x(t), u(t), h\big), \tag{7}$$

where $x \in R^s$, $f(x, u, h)$ is a function Lipschitz with respect to x and u, u is a bounded function, and $h > 0$ is a small parameter. Assume that system (7) is asymptotically stable for $u = 0$. Consider the set $\Omega = \{x : f(x, u, h = 0) = 0\}$. Then for each $\epsilon > 0$ there exists an $h_0 > 0$ such that the condition

$$\overline{\lim_{t\to\infty}}\,\mathrm{dist}\big(x(t), \Omega\big) < \varepsilon$$

holds for $h < h_0$.

The proof of this lemma can be found in [8, 15].

Let us verify the assumptions of Lemma 1 for system (6). System (6) is asymptotically stable for $z = 0$. Substituting $\mu = 0$ into (6), we obtain $G\xi = -Bz$, or $\xi_1 = z$ and $\xi_i = \xi_{i+1}$, $i = 1, \ldots, r-1$; that is, $\tilde{y} = z$. Then by Lemma 1 there exists a $\mu > 0$ such that the condition

$$\overline{\lim_{t\to\infty}}\big|\tilde{y}(t) - z(t)\big| < \delta_4 \tag{8}$$

is satisfied for $\mu \le \mu_0$, where $\delta_4 > 0$ is a sufficiently small number.

Consider the signal y consisting of the signal z and the noise w. Let us find a condition under which (5) is satisfied.

Taking into account (3) and (6), let us compose the discrepancy error $\zeta = \xi - \widetilde{\xi}$ in the form

$$\dot{\zeta}(t) = \frac{1}{\mu}G\zeta(t) + \frac{1}{\mu}Bw(t), \quad \hat{y}(t) - \tilde{y}(t) = L_r\zeta(t). \tag{9}$$

Let us find a solution of the first equation in (9) in the form

$$\zeta(t) = e^{\mu^{-1}Gt}\zeta(0) + \mu^{-1}\int\limits_0^t e^{\mu^{-1}G(t-s)}Bw(s)ds = \mu^{-1}\int\limits_0^t e^{\mu^{-1}G(t-s)}Bw(s)ds. \tag{10}$$

Under condition (4), it follows from (10) that

$$\overline{\lim_{t\to\infty}}\big|\zeta(t)\big| < \delta_2. \tag{11}$$

Consider the relations

$$\big|\hat{y}(t) - z(t)\big| \le \big|\hat{y}(t) - \tilde{y}(t)\big| + \big|\tilde{y}(t) - z(t)\big| \le \big|\zeta(t)\big| + \big|\tilde{y}(t) - z(t)\big|. \tag{12}$$

Obviously, the following passage to the limit holds for (12):

$$\overline{\lim_{t\to\infty}}\big|\hat{y}(t) - z(t)\big| \le \overline{\lim_{t\to\infty}}\big|\zeta(t)\big| + \overline{\lim_{t\to\infty}}\big|\tilde{y}(t) - z(t)\big|. \tag{13}$$

Let $\delta_2 + \delta_4 \le \delta_3$. Taking into account (8) and (11), we obtain the estimate (5) from (13). This completes the proof of Assertion 1. ■

Let the noise w be represented by the multiharmonic signal

$$w(t) = \sum_{i=1}^{v} A_i \sin(\omega_i t + \varphi_i), \tag{14}$$

where A_i, ω_i, and φ_i are the amplitude, frequency, and phase of the ith harmonic. For example, formula (14) can be obtained when expanding a bounded noise in a Fourier series with the subsequent extraction of a partial sum of the series. This technique is used, for example, to solve a number of problems in electrical engineering [3], computer science [6], control theory [2], and so on.

Assertion 2. *Let the assumptions be satisfied, let the function z be bounded, and let the noise w be represented by relation (14). Then there exists a $\mu_0 > 0$ such that for $\mu \le \mu_0$ one has*

$$\overline{\lim_{t\to\infty}}\big|\hat{y}(t) - z(t)\big| \le \sum_{i=1}^{v} A_i \left(\prod_{j=1}^{r}\frac{1}{\sqrt{1 + \omega_i^2\mu^2\sigma_j^2}}\right) + \delta_4. \tag{15}$$

Proof. It follows from the proof of Assertion 1 that for system (6) there exists a $\mu > 0$ such that condition (8) is satisfied for $\mu \le \mu_0$. Taking into account (3) and (6), we compose the discrepancy error $\zeta = \xi - \widetilde{\xi}$ in the form (9). Let us rewrite system (9) as

$$\begin{aligned} \dot{\zeta}_1(t) &= -\mu^{-1}\sigma_1^{-1}\zeta_1(t) + \mu^{-1}\sigma_1^{-1}\sum_{i=1}^{v} A_i \sin(\omega_i t + \varphi_i), \\ \dot{\zeta}_j(t) &= -\mu^{-1}\sigma_j^{-1}\zeta_j(t) + \mu^{-1}\sigma_j^{-1}\zeta_{j-1}(t), \quad j = 2, \dots, r. \end{aligned} \tag{16}$$

Let us find a solution of the first equation in system (16) in the form

$$\begin{aligned} \zeta_1(t) &= e^{-\mu^{-1}\sigma_1^{-1}t}\zeta_1(0) + \mu^{-1}\sigma_1^{-1}\sum_{i=1}^{v} A_i \int_0^t e^{\mu^{-1}\sigma_1^{-1}(s-t)} \sin(\omega_i s + \varphi_i)\, ds \\ &= \sum_{i=1}^{v} \frac{A_i}{\sqrt{1+\omega_i^2\mu^2\sigma_1^2}} \sin(\omega_i t + \varphi_i + \vartheta_{1,i}), \\ \vartheta_{1,i} &= \arccos\frac{1}{\sqrt{1+\omega_i^2\mu^2\sigma_1^2}}. \end{aligned} \tag{17}$$

Substituting (17) into the second equation in (16), we find a solution of the second equation in (16) in the form

$$\begin{aligned} \zeta_2(t) &= e^{-\mu^{-1}\sigma_2^{-1}t}\zeta_2(0) + \mu^{-1}\sigma_2^{-1}\sum_{i=1}^{v} \frac{A_i}{\sqrt{1+\omega_i^2\mu^2\sigma_1^2}} \int_0^t e^{\mu^{-1}\sigma_2^{-1}(s-t)} \sin(\omega_i s + \varphi_i + \vartheta_{1,i}) ds \\ &= \sum_{i=1}^{v} A_i \frac{1}{\sqrt{(1+\omega_i^2\mu^2\sigma_1^2)(1+\omega_i^2\mu^2\sigma_2^2)}} \sin(\omega_i t + \varphi_i + \vartheta_{1,i} + \vartheta_{2,i}), \\ \vartheta_{2,i} &= \arccos\frac{1}{\sqrt{1+\omega_i^2\mu^2\sigma_2^2}}. \end{aligned}$$

In a similar way, we find a solution of the last equation in system (16) in the form

$$\begin{aligned} \zeta_r(t) &= \sum_{i=1}^{v} A_i \left(\prod_{j=1}^{r} \frac{1}{\sqrt{1+\omega_i^2\mu^2\sigma_j^2}}\right) \sin(\omega_i t + \varphi_i + \sum_{j=1}^{r} \vartheta_{j,i}), \\ \vartheta_{j,i} &= \arccos\frac{1}{\sqrt{1+\omega_i^2\mu^2\sigma_j^2}}. \end{aligned} \tag{18}$$

Let us estimate (18) from above as

$$|\zeta_r(t)| \le \sum_{i=1}^{v} A_i \left(\prod_{j=1}^{r} \frac{1}{\sqrt{1+\omega_i^2\mu^2\sigma_j^2}}\right). \tag{19}$$

Taking into account relations (8), (13), and (19), we obtain the estimate (15). It follows from (19) that the larger ω_i, the smaller the right-hand side of (15) is. Therefore, a high-frequency noise is considered in the problem statement. The proof of Assertion 2 is complete. ■

To choose the parameters in the algorithm (3), one can use the following recommendations.

1. For the noises (14) with frequencies $\omega_i \geq \omega_0$. the parameters μ and σ_j specify the cutoff frequency of the filter (3) so that $\omega_0 \geq 1/(\mu\sigma_j)I$. Note that it is preferable to use $\omega_0 = 1/(\mu\sigma_j)$ for the calculation, since a low cutoff frequency can lead to the suppression of a useful part of the signal when forming the feedback.
2. The filter order r defines the slope of the amplitude-frequency response curve of the filter (3): a higher filter order provides stronger suppression of the noise beyond the cutoff frequency ω_0. Indeed, it follows from formula (19) that the noise amplitude A_i decreases by a factor of more than $2^{r/2}$.

4. CONTROL SYSTEM SYNTHESIS

For the subsequent synthesis of the control system, we use the algorithm in [9]. By the statement of the problem, the derivatives of $y(t)$ are inaccessible for measurement (Assumption 3). By virtue of the fact that the operator $Q(p)$ is normalized, in the control law we need to estimate the $(\gamma-1)$st derivative of the signal $\hat{y}(t)$. Hence we define the control law in the form

$$u(t) = -\alpha \sum_{i=0}^{\gamma-1} d_i \overline{y}^{(i)}(t), \tag{20}$$

where $\alpha > 0$, the coefficients $d_0, d_1, \ldots, d_{\gamma-1}$ are selected so that the polynomial $D(\lambda) = d_{\gamma-1}\lambda^{\gamma-1} + d_{\gamma-2}\lambda^{\gamma-2} + \cdots d_1\lambda + d_0$ is Hurwitz, $\gamma = n - m$ is the plant relative degree, and $\overline{y}^{(i)}(t)$ is an estimate of the ith derivative of the signal $\hat{y}(t)$, $i = 0, 1, \ldots, \gamma - 1$.

Substituting (20) into (1), we obtain

$$F(p)z(t) = \alpha R(p)g(t) + \alpha R(p)D(p)\psi(t) + f(t), \tag{21}$$

where $F(p) = Q(p) + \alpha R(p)D(p)$, $g(t) = D(p)\hat{y}(t) - \sum_{i=0}^{\gamma-1} d_i \overline{y}^{(i)}(t)$ is the signal characterizing the deviation of the ith derivative of the "filtered" plant output $\hat{y}(t)$ from the corresponding estimate $\overline{y}^{(i)}(t)$, and $\psi(t) = z(t) - \hat{y}(t)$. The value of the function $g(t)$ depends on the quality of estimation of the derivatives of the signal $\hat{y}(t)$, and the value of the function $\psi(t)$ depends on the quality of filtering of the algorithm (3). Since the set Ξ is known (Assumption 1), there always exists a number α and a polynomial $D(\lambda)$ such that the polynomial $F(\lambda)$ is Hurwitz; see [1, 4, 7]. We point out that the polynomial $D(\lambda)$ is chosen so that the operating frequencies of the closed-loop control system (21) are in the passband of the filter (3). Such a choice is always feasible, because system (21) possesses the property of a low-pass filter ($\gamma \geq 1$).

To implement the control law (20), we use the observer defined by the equations

$$\begin{aligned} \overline{y}(t) &= \hat{y}(t), \\ \overline{y}^{(j)}(t) &= \frac{\overline{y}^{(j-1)}(t) - \overline{y}^{(j-1)}(t-h)}{h}, \quad j = 1, \ldots, \gamma - 1. \end{aligned} \tag{22}$$

Substituting (22) into (20), we rewrite the control law (20) in the form

$$u(t) = -\alpha \sum_{i=0}^{\gamma-1} \left[\frac{d_i}{h^i} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \hat{y}(t - jh) \right], \tag{23}$$

where $\binom{i}{j} = \frac{i!}{j!\cdot(i-j)!}$ is the binomial coefficient. Let us state an assertion with conditions that guarantee the validity of the target condition (2).

Assertion 3. *Let the conditions in Assumptions 1-3 be satisfied, and let the function w be bounded if $r \geq \gamma$ and bounded together with its $(\gamma - r - 1)$ derivatives if $r < \gamma$. Then there exist*

numbers $\alpha > 0$ and $h > 0$ such that the control system represented by the algorithm (3) and the control law (23) ensures the validity of the target condition (2) and the boundedness of all signals in the closed-loop system.

Proof. Let us transform system (21) to the form

$$\dot{\varepsilon}(t) = A\varepsilon(t) + \alpha B_1 g(t) + \alpha B_2 \psi(t) + B_3 f(t), \quad z(t) = J_n \varepsilon(t), \tag{24}$$

where $\epsilon = [\epsilon_1, \epsilon_2, \ldots, \epsilon_n]^{\mathrm{T}}$, $z^{(i)} = \epsilon_{i+1}$, $i = 0, \ldots, n-1$, and the matrices A, B_1, B_2, and B_3 have been obtained in the transition from (21) to (24). Here and in the following, the matrix $J_l = [1, 0, \ldots, 0]$ has the size $1 \times l$. For example, the size of J_n in (24) is $1 \times n$.

For convenience of the subsequent study of the stability of the closed-loop system with the use of the technique of Lyapunov–Krasovskii functionals, we rewrite system (3) in the canonical observable form

$$\dot{\theta}(t) = M\theta(t) + Ny(t), \quad \hat{y}(t) = J_r \theta(t), \tag{25}$$

where $\theta = [\theta_1, \ldots, \theta_r]^{\mathrm{T}}$, $\hat{y}^{(i)} = \theta_{i+1}$, $i = 0, \ldots, r-1$, and the matrix $M = \mu^{-1} TGT^{-1}$ and the vector $N = \mu^{-1} TB$ have been obtained in the transition from (3) to (25) using the transformation matrix $T = [C^{\mathrm{T}}, (AC)^{\mathrm{T}}, \ldots, (A^{r-1}C)^{\mathrm{T}}]^{\mathrm{T}}$.

Consider two cases depending on the dimension of the filter (3).

1. Let $r < \gamma$. Rewrite the operator $D(p)$ in the form

$$D(p) = \rho_1^{\mathrm{T}} \begin{bmatrix} p^{\gamma-1} & p^{\gamma-2} & \ldots & p^r \end{bmatrix} + \rho_2^{\mathrm{T}} \begin{bmatrix} p^{r-1} & p^{r-2} & \ldots & 1 \end{bmatrix}, \tag{26}$$

where ρ_1 and ρ_2 are the vectors with the corresponding coefficients of the operator $D(p)$. In view of (26), we rewrite the function $g(t)$ in the form

$$g(t) = \sum_{j=1}^{\gamma-r} \rho_{1,(j-1+r)} L_r \theta^{(j)} + \rho_2^T \theta(t) - \sum_{i=0}^{\gamma-1} \frac{d_i}{h^i} J_r \theta(t) - \sum_{i=1}^{\gamma-1} \left[\frac{d_i}{h^i} \sum_{j=1}^{i} (-1)^j \binom{i}{j} J_r \theta(t - jh) \right], \tag{27}$$

where $\rho_{1,j}$ is the jth component of the vector ρ_1. Taking into account (27), we find the jth derivative $(1 \le j \le n)$ of θ in the form

$$\begin{aligned} \theta^{(j)} &= M^j \theta + \sum_{i=0}^{j-1} M^{j-i-1} N y^{(i)} \\ &= M^j \theta + \sum_{i=0}^{j-1} M^{j-i-1} N z^{(i)} + \sum_{i=0}^{j-1} M^{j-i-1} N w^{(i)} \\ &= M^j \theta + \sum_{i=0}^{j-1} M^{j-i-1} N K_{i+1} \varepsilon + \sum_{i=0}^{j-1} M^{j-i-1} N w^{(i)}, \end{aligned} \tag{28}$$

where $K_j = \begin{bmatrix} 0, \ldots, 0, 1, 0, \ldots, 0 \end{bmatrix}$ is the matrix of size $1 \times n$ with zero entries except for the jth one, which is equal to 1.

Substituting (27) into (28), we obtain

$$\begin{aligned} g = &\sum_{j=1}^{\gamma-r} \rho_{1,(j-1+r)} L_r \left(M^j \theta + \sum_{i=0}^{j-1} M^{j-i-1} N K_{i+1} \varepsilon + \sum_{i=0}^{j-1} M^{j-i-1} N w^{(i)} \right) \\ &+ \rho_2^{\mathrm{T}} \theta(t) - \sum_{i=0}^{\gamma-1} \frac{d_i}{h^i} J_r \theta(t) - \sum_{i=1}^{\gamma-1} \left[\frac{d_i}{h^i} \sum_{j=1}^{i} (-1)^j \binom{i}{j} J_r \theta(t - jh) \right]. \end{aligned} \tag{29}$$

It follows from (29) that for $r < \gamma$ it is required that the derivatives order $\le \gamma - r - 1$ of the function w be bounded. In view of (23) and (29), we transform Eq. (24) into

$$\dot\varepsilon(t) = A\varepsilon(t) + \alpha B_1 \left(\sum_{j=1}^{\gamma-r} \rho_{1,(j-1+r)} L_r \left(M^j\theta + \sum_{i=0}^{j-1} M^{j-i-1} N K_{i+1}\varepsilon + \sum_{i=0}^{j-1} M^{j-i-1} N w^{(i)} \right) + \rho_2^{\mathrm T}\theta(t) - \sum_{i=0}^{\gamma-1} \frac{d_i}{h^i} J_r\theta(t) \right) - \alpha B_1 \left[\sum_{i=1}^{\gamma-1} \frac{d_i}{h^i} \sum_{j=1}^{i} (-1)^j \binom{i}{j} J_r\theta(t-jh) \right] + \alpha B_2 J_n\varepsilon(t) - \alpha B_2 J_r\theta(t) + B_3 f(t). \tag{30}$$

Let us introduce the notation

$$\varepsilon_p(t) = \left[\varepsilon^{\mathrm T}(t), \quad \theta^{\mathrm T}(t)\right]^{\mathrm T},$$

$$A_{11} = A + \alpha B_1 \sum_{j=1}^{\gamma-r} \rho_{1,(j-1+r)} L_r \sum_{i=0}^{j-1} M^{j-i-1} N K_{i+1} + \alpha B_2 J_n,$$

$$A_{12} = \alpha B_1 \left(\sum_{j=r}^{\gamma-r} \rho_{1,(j-1+r)} L_r M^j + \rho_2^{\mathrm T} - \sum_{i=0}^{\gamma-1} \frac{d_i}{h^i} J_r \right) - \alpha B_2 J_r,$$

$$A_p = \begin{bmatrix} A_{11} & A_{12} \\ NJ_n & M \end{bmatrix}, \quad F_{ij} = \begin{bmatrix} 0 & (-1)^{j+1} \alpha B_1 \frac{d_i}{h^i} \binom{i}{j} J_r \\ 0 & 0 \end{bmatrix},$$

$$i = 1, \dots, \gamma - 1, \quad j = 1, \dots, i,$$

$$\vartheta = \begin{bmatrix} 0 \\ N \end{bmatrix} w + \sum_{j=1}^{\gamma-r} \sum_{i=0}^{j-1} \begin{bmatrix} \alpha B_1 \rho_{1,(j-1+r)} L_r M^{j-i-1} N \\ 0 \end{bmatrix} w^{(i)} + \begin{bmatrix} B_3 \\ 0 \end{bmatrix} f.$$

Here ϑ is a bounded function. Using this notation, we rewrite systems (28) and (30) in the form

$$\dot\varepsilon_p(t) = A_p\varepsilon_p(t) + \sum_{i=1}^{\gamma-1} \sum_{j=1}^{i} F_{ij}\varepsilon_p(t-jh) + \vartheta(t). \tag{31}$$

Consider a Lyapunov–Krasovskii functional of the form

$$V = \varepsilon_p^{\mathrm T}(t) P \varepsilon_p(t) + \sum_{i=1}^{\gamma-1} \sum_{j=1}^{i} \int_{-jh}^{0} \varepsilon_p^{\mathrm T}(t+s) N_{ij} \varepsilon_p(t+s) ds, \tag{32}$$

where $P = P^{\mathrm T} > 0$ is a solution of the equation $A_p^{\mathrm T} P + P A_p = -Q$, $Q = Q^{\mathrm T} > 0$, $N_{ij} = N_{ij}^{\mathrm T} > 0$. Taking the time derivative of the functional (32) along the trajectories of system (31), we obtain

$$\dot V = -\varepsilon_p^{\mathrm T}(t) Q \varepsilon_p(t) + 2\varepsilon_p^{\mathrm T}(t) P \sum_{i=1}^{\gamma-1} \sum_{j=1}^{i} F_{ij}\varepsilon_p(t-jh) + 2\varepsilon_p^{\mathrm T}(t) P \vartheta(t) + \sum_{i=1}^{\gamma-1} \sum_{j=1}^{i} \left(\varepsilon_p^{\mathrm T}(t) N_{ij} \varepsilon_p(t) - \varepsilon_p^{\mathrm T}(t-jh) N_{ij} \varepsilon_p(t-jh) \right). \tag{33}$$

Let us estimate the terms in (33) from above,

$$2\varepsilon_p^{\mathrm{T}}(t)P\sum_{i=1}^{\gamma-1}\sum_{j=1}^{i}F_{ij}\varepsilon_p(t-jh)\le 0.5\gamma\left(\gamma-1\right)\chi\varepsilon_p^{\mathrm{T}}(t)P^2\varepsilon_p(t)$$
$$+\chi^{-1}\sum_{i=1}^{\gamma-1}\sum_{j=1}^{i}\varepsilon_p^{\mathrm{T}}(t-jh)F_{ij}^{\mathrm{T}}F_{ij}\varepsilon_p(t-jh),$$
$$2\varepsilon_p^{\mathrm{T}}(t)P\vartheta(t)\le\chi\varepsilon_p^{\mathrm{T}}(t)P^2\varepsilon_p(t)+\chi^{-1}\left|\vartheta(t)\right|^2.$$

Taking these estimates into account, we rewrite (30) in the form

$$\dot V\le-\varepsilon_p^{\mathrm{T}}(t)W\varepsilon_p(t)-\sum_{i=1}^{\gamma-1}\sum_{j=1}^{i}\varepsilon_p^{\mathrm{T}}(t-jh)R_{ij}\varepsilon_p(t-jh)+\chi^{-1}\tau,\tag{34}$$

where

$$W=Q-0.5\gamma\left(\gamma-1\right)\chi P^2-\chi P^2-\sum_{i=1}^{\gamma-1}\sum_{j=1}^{i}N_{ij},\quad R_{ij}=N_{ij}-\chi^{-1}F_{ij}^{\mathrm{T}}F_{ij},\quad\tau=\sup_t\left|\vartheta(t)\right|^2.$$

Obviously, there exists an α and a χ such that $W>0$ and $R_{ij}>0$. Let us estimate (34) in the form

$$\dot V\le-\lambda_{\min}(W)\varepsilon_p^{\mathrm{T}}(t)\varepsilon_p(t)+\chi^{-1}\tau.\tag{35}$$

Here $\lambda_{\min}(W)$ is the least eigenvalue of the matrix W. It follows from (35) that $|z(t)|\le|\varepsilon_p(t)|\le\sqrt{\chi^{-1}\tau/\lambda_{\min}(W)}$. Consequently, by appropriately choosing α and χ so that $W>0$ and $R_{ij}>0$, we can ensure that the condition $\delta_1\le\sqrt{\chi^{-1}\tau/\lambda_{\min}(W)}$ is satisfied. It is worth mentioning that the estimate for z is rather coarse, because in the proof we have used coarse estimates and from (35) we have eliminated the terms $\sum_{i=1}^{\gamma-1}\sum_{j=1}^{i}\varepsilon_p^{\mathrm{T}}(t-jh)R_{ij}\varepsilon_p(t-jh)$ compensating for $\chi^{-1}\tau$.

2. Let $r\ge\gamma$. Rewrite the operator $D(p)$ in the form

$$D(p)=\rho^{\mathrm{T}}\begin{bmatrix}p^{\gamma-1}&p^{\gamma-2}&\dots&1\end{bmatrix}.$$

Taking into account (23), we rewrite the function $g(t)$ in the form

$$g(t)=\rho^{\mathrm{T}}\theta(t)-\sum_{i=0}^{\gamma-1}\frac{d_i}{h^i}J_r\theta(t)-\sum_{i=1}^{\gamma-1}\left[\frac{d_i}{h^i}\sum_{j=1}^{i}(-1)^j\binom{i}{j}J_r\theta(t-jh)\right].\tag{36}$$

In view of (36), we reduce Eq. (24) to the form

$$\dot\varepsilon(t)=A\varepsilon(t)+\alpha kB_1\left(\rho^{\mathrm{T}}\theta(t)-\sum_{i=0}^{\gamma-1}\frac{d_i}{h^i}J_r\theta(t)\right)-\alpha kB_1\left[\sum_{i=1}^{\gamma-1}\frac{d_i}{h^i}\sum_{j=1}^{i}(-1)^j\binom{i}{j}J_r\theta(t-jh)\right]$$
$$+\alpha kB_2J_n\varepsilon(t)-\alpha kB_2J_r\theta(t)+B_3f(t).\tag{37}$$

Rewrite systems (28) and (37) in the form

$$\dot\varepsilon_p(t)=A_p\varepsilon_p(t)+\sum_{i=1}^{\gamma-1}\sum_{j=1}^{i}F_{ij}\varepsilon_p(t-jh)+\vartheta(t),\tag{38}$$

Table 1. Maximum value of errors in estimating derivatives of the signal $z(t)$ in a steady state when using the proposed scheme as well as the schemes in [14] and [12, 22]

Control system type	Value of $\sup\limits_t \lvert z(t) - \overline{y}(t)\rvert$ 8 s after the start of system operation	Value of $\sup\limits_t \lvert \dot{z}(t) - \overline{y}^{(1)}(t)\rvert$ 8 s after the start of system operation	Value of $\sup\limits_t \lvert \ddot{z}(t) - \overline{y}^{(2)}(t)\rvert$ 8 s after the start of system operation	Value of $\sup\limits_t \lvert \dddot{z}(t) - \overline{y}^{(3)}(t)\rvert$ 8 s after the start of system operation
Control system (40), (43) (based on the high-gain observer [14])	0.22	8.41	132.3	698
Control system (40), (42) (based on the high-gain observer under measurement noise conditions [12, 22])	0.26	4.9	31.1	266.1
Control system (3), (40), (41) with $r = 2$ in (3)	0.04	0.2	2	57
Control system (3), (40), (41) with $r = 5$ in (3)	$3 \cdot 10^{-3}$	$7.5 \cdot 10^{-4}$	$2 \cdot 10^{-3}$	$5 \cdot 10^{-3}$

where

$$A_p = \begin{bmatrix} A + \alpha B_2 J_n & \alpha B_1 \left(\rho^{\mathrm{T}} - \sum_{i=0}^{\gamma-1} \frac{d_i}{h^i} J_r\right) - \alpha B_2 J_r \\ N J_n & M \end{bmatrix},$$

$\vartheta = \begin{bmatrix} 0 \\ N \end{bmatrix} w + \begin{bmatrix} B_3 \\ 0 \end{bmatrix} f$ is a bounded function, and the structure of the matrix F_{ij} corresponds to that of the matrix $\bar{F}_{ij}$ in (31).

Since the structure of system (38) is similar to that of system (31), wse see that the subsequent proof of the second case is similar to that of the first case. The proof of Assertion 3 is complete. ■

5. EXAMPLE

Consider a plant of the form

$$\begin{aligned} \left(p^4 + q_3 p^3 + q_2 p^2 + q_1 p + q_0\right) z(t) &= u(t) + f(t), \\ y(t) &= z(t) + w(t). \end{aligned} \tag{39}$$

The set Ξ of possible parameter values in (39) is given by the inequalities

$$-1 \le q_3 \le 0.1; \quad -2 \le q_2 \le 2; \quad -3 \le q_1 \le 3; \quad -1 \le q_0 \le 1.$$

In addition, $|f(t)| \le 1$.

Let us select $\sigma_i = 1$ and $\mu = 0.01$ in (3). The order r in the algorithm (3) will be defined in Table 1.

Let $\alpha = 7$, $d_0 = 0.9$, $d_1 = 1.5$, $d_2 = 2$, and $d_3 = 0.5$. Then the control law (23) can be written in the form

$$u(t) = -7\left(0.9\overline{y}(t) + 1.5\overline{y}^{(1)}(t) + 2\overline{y}^{(2)}(t) + 0.5\overline{y}^{(3)}(t)\right). \tag{40}$$

To estimate the derivatives in (40), we use the observer (22). Let $h = 1/20$. Then we rewrite the observer (22) in the form

$$\begin{aligned}\overline{y}(t) &= \hat{y}(t),\\ \overline{y}^{(1)}(t) &= 20\big[\overline{y}(t) - \overline{y}(t-0.05)\big],\\ \overline{y}^{(2)}(t) &= 20\big[\overline{y}^{(1)}(t) - \overline{y}^{(1)}(t-0.05)\big],\\ \overline{y}^{(3)}(t) &= 20\big[\overline{y}^{(2)}(t) - \overline{y}^{(2)}(t-0.05)\big].\end{aligned} \tag{41}$$

Let us demonstrate the efficiency of the proposed scheme (3), (40), and (41) compared with the high-gain observer in [3] and the high-gain observer under measurement noise conditions in [15, 16]. To this end, we again define a control law in the form (40) but use the following observers for estimating the derivatives in (40):

1. The high-gain observer in [14],

$$\begin{aligned}\dot{\xi}(t) &= \begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\xi(t) + \begin{bmatrix}110\\110^2\cdot 0.35\\110^3\cdot 0.05\\110^4\cdot 0.0024\end{bmatrix}\Big(y(t) - \begin{bmatrix}1&0&0&0\end{bmatrix}\xi(t)\Big),\\ \overline{y}(t) &= \begin{bmatrix}1&0&0&0\end{bmatrix}\xi(t), \quad \overline{y}^{(1)}(t) = \begin{bmatrix}0&1&0&0\end{bmatrix}\xi(t),\\ \overline{y}^{(2)}(t) &= \begin{bmatrix}0&0&1&0\end{bmatrix}\xi(t), \quad \overline{y}^{(3)}(t) = \begin{bmatrix}0&0&0&1\end{bmatrix}\xi(t).\end{aligned} \tag{42}$$

2. The high-gain observer under measurement noise conditions in [12, 22],

$$\begin{aligned}\dot{\eta}_1(t) &= \begin{bmatrix}0&1\\0&0\end{bmatrix}\eta_1(t) + \begin{bmatrix}0&0\\0&1\end{bmatrix}\eta_2(t)\\ &\quad + \begin{bmatrix}110&0\\0&110^2\end{bmatrix}\begin{bmatrix}0.5\\0.16\end{bmatrix}\Big(y(t) - \begin{bmatrix}1&0\end{bmatrix}\eta_1(t)\Big),\\ \dot{\eta}_2(t) &= \begin{bmatrix}0&1\\0&0\end{bmatrix}\eta_2(t) + \begin{bmatrix}0&0\\0&1\end{bmatrix}\eta_3(t)\\ &\quad + \begin{bmatrix}110&0\\0&110^2\end{bmatrix}\begin{bmatrix}0.5\\0.0525\end{bmatrix}\Big(\begin{bmatrix}0&1\end{bmatrix}\eta_1(t) - \begin{bmatrix}1&0\end{bmatrix}\eta_2(t)\Big),\\ \dot{\eta}_3(t) &= \begin{bmatrix}0&1\\0&0\end{bmatrix}\eta_3(t)\\ &\quad + \begin{bmatrix}110&0\\0&110^2\end{bmatrix}\begin{bmatrix}0.5\\0.0171\end{bmatrix}\Big(\begin{bmatrix}0&1\end{bmatrix}\eta_2(t) - \begin{bmatrix}1&0\end{bmatrix}\eta_3(t)\Big),\\ \overline{y}(t) &= \begin{bmatrix}1&0\end{bmatrix}\eta_1(t), \quad \overline{y}^{(1)}(t) = \begin{bmatrix}1&0\end{bmatrix}\eta_2(t),\\ \overline{y}^{(2)}(t) &= \begin{bmatrix}1&0\end{bmatrix}\eta_3(t), \quad \overline{y}^{(3)}(t) = \begin{bmatrix}0&1\end{bmatrix}\eta_3(t).\end{aligned} \tag{43}$$

Let $q_3 = 0$, $q_2 = 1$, $q_1 = 1$, $q_0 = 0$, $f(t) = 0$, $w(t) = \sin(0.5\cdot 10^3 t)$ and $z(0) = 1$, $\dot{z}(0) = \ddot{z}(0) = 0$, $\dddot{z}(0) = -1$ in (39). Table 1 lists the maximum values of the estimation errors for the derivatives

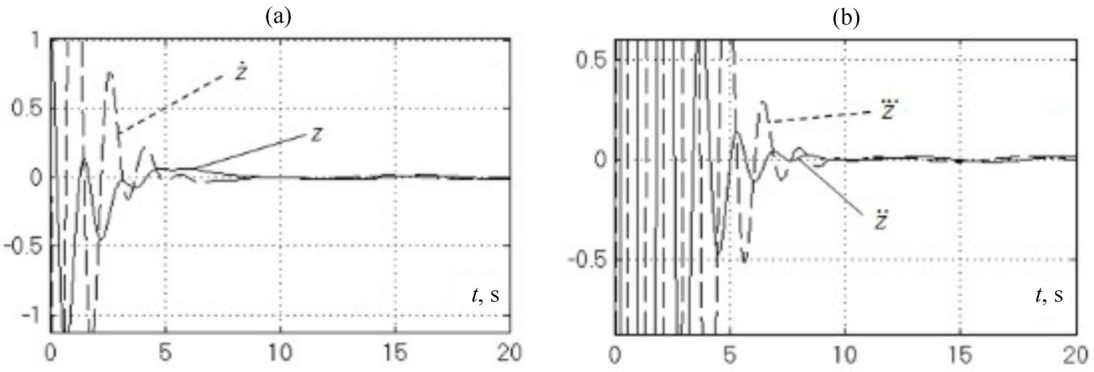


Fig. 1. Simulation results for (a) $z(t)$, $\dot{z}(t)$ and (b) $\ddot{z}(t)$, $\dddot{z}(t)$.

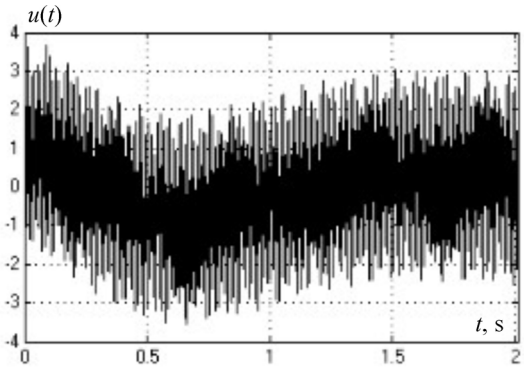


Fig. 2. Simulation results for $u(t)$.

of the signal $z(t)$ in the steady state when using the control system (3), (40), (41), the control system (40), (42) (based on [14]), and the control system (40), (43) (based on [12, 22]).

It can be seen from Table 1 that the proposed control algorithm permits considerably reducing the errors in the estimation of the derivatives of the signal z, and for $r = 2$ the dynamic order of the proposed algorithm is one less than that of the algorithm in [12, 22]. In this case, it follows from Table 1 that by increasing the parameter r one can improve the derivative estimation quality.

Let $q_3 = 0.1$, $q_2 = 2$, $q_1 = 3$, $q_0 = 1$, $f(t) = \sin t$, $w(t) = \sin(0.5 \cdot 10^3 t) + \sin(10^3 t) + \sin(10^4 t)$, $z(0) = 1$, $\dot{z}(0) = \ddot{z}(0) = 0$, $\dddot{z}(0) = -1$ in (3). Figure 1 shows the results of modeling for $z(t)$, $\dot{z}(t)$, $\ddot{z}(t)$, and $\dddot{z}(t)$ when using the control system (3), (40), (41) with $r = 5$ in (3). The graphs of $z(t)$ and $\ddot{z}(t)$ are shown by solid curves and the graphs of $\dot{z}(t)$ and $\dddot{z}(t)$, by dashed curves.

The results of modeling showed (Fig. 1) that after 10 s the absolute values of the signals $z(t)$, $\dot{z}(t)$, $\ddot{z}(t)$, and $\dddot{z}(t)$ do not exceed 0.014, and the absolute values of the estimation errors for the signals $z(t)$, $\dot{z}(t)$, $\ddot{z}(t)$, and $\dddot{z}(t)$ are under 2.10^{-3}. The modeling results for the control are shown in Fig. 2. We point out that the transients in $u(t)$ for each of the schemes [12, 14, 22] are similar. However, given the same level and quality of the control signal, different quality indicators with respect to the output signal were obtained (see Table 1), and these indicators assume the least value for the proposed control algorithm.

6. CONCLUSIONS

The article synthesizes a robust control algorithm under conditions of parametric uncertainty, external bounded disturbances, and high-frequency noise in the measurements of the output variable. For the synthesis of the control system, we have an approach that permits independently controlling

the noise filtering quality and the quality of the output-variable stabilization error. Simulation results are presented that illustrate the efficiency of our algorithm in comparison with the classical high-gain observer [14] and the high-gain observer under measurement noise conditions [12, 22].

FUNDING

The results in Sec. 3 were supported by a grant from the President of the Russian Federation, project no. MD-1054.2020.8, agreement no. 075-15-2020-184 at the Institute for Problems in Mechanical Engineering of the Russian Academy of Sciences. The results in 4 were produced at the Institute for Problems in Mechanical Engineering of the Russian Academy of Sciences and supported by the Russian Science Foundation, project no. 18-79-10104.

REFERENCES

1. Bobtsov, A.A., Robust output-control for a linear system with uncertain coefficients, *Autom. Remote Control*, 2002, no. 11, pp. 1794–1802.
2. Voronov, A.A., *Teoriya avtomaticheskogo upravleniya. Chast' 2. Teoriya nelineinykh i spetsial'nykh sistem avtomaticheskogo upravleniya* (Automated Control Theory. Part 2. Theory of Nonlinear and Special Automated Control Systems), Moscow: Vyssh. Shkola, 1986.
3. Zhezhelenko, I.V., *Vysshie garmoniki v sistemakh elektrosnabzheniya prompredpriyatii* (Higher Harmonics in Industrial Power Supply Systems), Moscow: Energoatomizdat, 1984.
4. Miroshnik, I.V., Nikiforov, V.O., and Fradkov, A.L., *Nelineinoe i adaptivnoe upravlenie slozhnymi dinamicheskimi sistemami* (Nonlinear and Adaptive Control of Complex Dynamical Systems), St. Petersburg: Nauka, 2000.
5. Polyak, B.T. and Shcherbakov, P.S., *Robastnaya ustoichivost' i upravlenie* (Robust Stability and Control), Moscow: Nauka, 2002.
6. Fink, L.M., *Teoriya peredachi diskretnykh soobshchenii* (Discrete Message Transmission Theory), Moscow: Sov. Radio, 1970.
7. Fradkov, A.L., Synthesis of an adaptive stabilization system for a linear dynamic plant, *Avtom. Telemekh.*, 1974, no. 12, pp. 96–103.
8. Furtat, I.B., Robust control for a specific class of non-minimum phase dynamical networks, *J. Comput. Syst. Sci. Int.*, 2014, vol. 53, no. 1, pp. 33–46.
9. Furtat, I.B., Robust static control algorithm for linear objects, *Autom. Remote Control*, 2015, vol. 76, no. 3, pp. 446–457.
10. Tsykunov, A.M., An algorithm of robust control of a non-stationary linear plant with perturbation compensation, *J. Comput. Syst. Sci. Int.*, 2008, vol. 47, no. 4, pp. 527–534.
11. Ahrens, J. and Khalil, K., High-gain observers in the presence of measurement noise: a switched-gain approach, *Automatica*, 2009, vol. 45, pp. 936–943.
12. Astolfi, D. and Marconi, L., A high-gain nonlinear observer with limited gain power, *IEEE Trans. Autom. Control*, 2015, vol. 60, no. 11, pp. 3059–3064.
13. Boizot, N., Busvelle, E., and Gauthier, J., An adaptive high-gain observer for nonlinear systems, *Automatica*, 2010, vol. 46, pp. 1483–1488.
14. Esfandiari, F. and Khalil, H.K., Output feedback stabilization of fully linearizable systems, *Int. J. Control*, 1992, vol. 56, no. 5, pp. 1007–1037.
15. Furtat, I., Fradkov, A., and Tsykunov, A., Robust synchronization of linear dynamical networks with compensation of disturbances, *Int. J. Robust Nonlinear Control*, 2014, vol. 24, no. 17, pp. 2774–2784.
16. Furtat, I.B., Fradkov, A.L., and Liberzon, D., Compensation of disturbances for MIMO systems with quantized output, *Automatica*, 2015, vol. 60, pp. 239–244.

17. Gauthier, J., Hammouri, H., and Othman, S., A simple observer for nonlinear systems application to bioreactors, *IEEE Trans. Autom. Control*, 1992, vol. 37, no. 6, pp. 875–880.

18. Prasov, A. and Khalil, H., A nonlinear high-gain observer for systems with measurement noise in a feedback control framework, *IEEE Trans. Autom. Control*, 2013, vol. 58, no. 3, pp. 569–580.

19. Sanfelice, R. and Praly, L., On the performance of high-gain observers with gain adaptation under measurement noise, *Automatica*, 2011, vol. 47, pp. 2165–2176.

20. Teel, A.R. and Praly, L., Tools for semiglobal stabilization by partial state and output feedback, *SIAM J. Control Optim.*, 1994, vol. 33, no. 5, pp. 1443–1488.

21. Vasiljevic, L. and Khalil, H., Error bounds in differentiation of noisy signals in high-gain observers, *Syst. Control Lett.*, 2008, vol. 57, pp. 856–862.

22. Wang, L., Astolfi, D., Hongye, S., Marconi, L., and Isidori, A., Output stabilization for a class of nonlinear systems via high-gain observer with limited gain power, *Proc. 1st IFAC Conf. Model. Ident. Control Nonlinear Syst., MICNON 2015* (St. Petersburg, Russia, 2015), *IFAC-PapersOnLine,* vol. 48, no. 11, pp. 730–735.