

For problems 1–4, we are working with $f(x) = e^x$. We'll use different approximations of f near 0 to approximate the numbers $e^{0.1}$ and e .

1. **Linear Approximation:** We want to find a linear function $L(x) = C_0 + C_1 x$ whose value and slope at $x = 0$ matches the value and slope of f at $x = 0$.

(a) Fill in the table below:

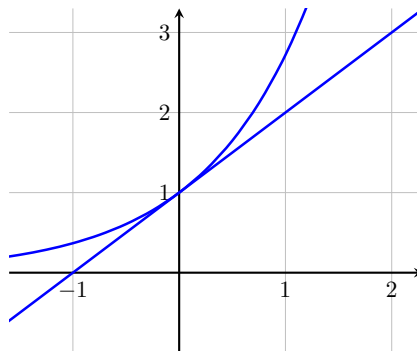
$f(x) = e^x$	$f(0) = e^0 = 1$	$L(x) = C_0 + C_1 x$	$L(0) = C_0 + 0 = C_0$
$f'(x) = e^x$	$f'(0) = 1$	$L'(x) = C_1$	$L'(0) = C_1$

(b) Find the coefficients of L .

- If $f(x)$ and $L(x)$ have the same value at 0, then $f(0) = L(0)$ so $C_0 = \underline{1}$
- If $f(x)$ and $L(x)$ have the same slope at 0, then $f'(0) = L'(0)$ so $C_1 = \underline{1}$

Thus $L(x) = \underline{1 + x}$

(c) Graph e^x and $L(x)$.



(d) Use $L(x)$ to estimate $e^{0.1}$ and e .

$$e^{0.1} = f(0.1) \approx L(0.1) = 1 + 0.1 = 1.1$$

$$e^1 = f(1) \approx L(1) = 1 + 1 = 2$$

(e) Plug $e^{0.1}$ and e into a calculator and compare these values to the estimates you got in part (d). How accurate are your estimates?

Solution: A calculator gives $e^{0.1} \approx 1.1051...$ so $0.005 < |\text{error}| < 0.006$

A calculator gives $e^1 \approx 2.718...$, so $0.7 < |\text{error}| < 0.8$ which is much worse. This is because linear approximation is centered at 0, so our approximation gets worse the farther we get from 0.

Note: The general formula for a linear approximation centered at $x = 0$ is $L(x) = f(0) + f'(0)x$. This is the equation of the tangent line to f at 0. It is also called the **1st degree Taylor Polynomial** for $f(x)$ centered at $x = 0$. Confirm that the answer you got for $L(x)$ in part (b) matches this formula.

2. **Quadratic Approximation:** We want to find a quadratic function $Q(x) = C_0 + C_1x + C_2x^2$ whose value, slope and concavity at $x = 0$ matches that of f at $x = 0$.

(a) Fill in the table below.

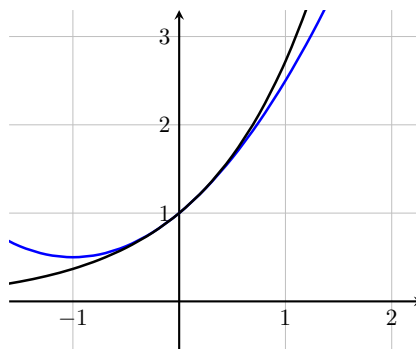
$f(x) = e^x$	$f(0) = 1$	$Q(x) = C_0 + C_1x + C_2x^2$	$Q(0) = C_0$
$f'(x) = e^x$	$f'(0) = 1$	$Q'(x) = C_1 + 2C_2x$	$Q'(0) = C_1$
$f''(x) = e^x$	$f''(0) = 1$	$Q''(x) = 2C_2$	$Q''(0) = 2C_2$

(b) Find the coefficients of Q .

- If $f(x)$ and $Q(x)$ have the same value at 0, then $f(0) = Q(0)$ so $C_0 = \underline{1}$
- If $f(x)$ and $Q(x)$ have the same slope at 0, then $f'(0) = Q'(0)$ so $C_1 = \underline{1}$
- If $f(x)$ and $Q(x)$ have the same concavity at 0, then $f''(0) = Q''(0)$ so $2C_2 = \underline{1}$, hence $C_2 = \underline{\frac{1}{2}}$

Thus $Q(x) = \underline{1 + x + \frac{1}{2}x^2}$

(c) A graph of e^x is shown below. Graph $Q(x)$ on the same axes.



(d) Use $Q(x)$ to estimate $e^{0.1}$ and e .

(e) Compare the estimates you got in part (d) to the values you got from your calculator. Explain why the errors are reduced.

Note: The general formula for a quadratic approximation of f centered at $x = 0$ (AKA the **2nd degree Taylor polynomial** of f centered at $x = 0$) is:

$$Q(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2.$$

Check that your answer for $Q(x)$ in part (b) matches this formula.

3. **Cubic Approximation:** We want to find a cubic function $T(x) = C_0 + C_1x + C_2x^2 + C_3x^3$ whose value and whose first, second and third derivatives at $x = 0$ match those of f at $x = 0$.

(a) Fill in the table below:

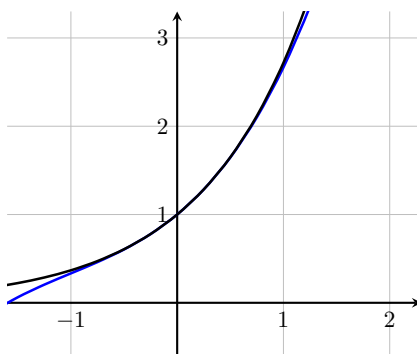
$f(x) = e^x$	$f(0) = 1$	$T(x) = C_0 + C_1x + C_2x^2 + C_3x^3$	$T(0) = C_0$
$f'(x) = e^x$	$f'(0) = 1$	$T'(x) = C_1 + 2C_2x + 3C_3x^2$	$T'(0) = C_1$
$f''(x) = e^x$	$f''(0) = 1$	$T''(x) = 2C_2 + 3 \cdot 2x$	$T''(0) = 2C_2$
$f'''(x) = e^x$	$f'''(0) = 1$	$T'''(x) = 3 \cdot 2C_3$	$T'''(0) = 6C_3$

(b) Find the coefficients of T .

- If $f(0) = T(0)$, then $C_0 = \underline{1}$
- If $f'(0) = T'(0)$, then $C_1 = \underline{1}$
- If $f''(0) = T''(0)$, then $\underline{2C_2 = 1}$
- If $f'''(0) = T'''(0)$, then $\underline{6C_3 = 1}$

Thus $T(x) = \underline{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3}$

(c) A graph of e^x is shown below. Graph $T(x)$ on the same coordinate plane.



(d) Use $T(x)$ to estimate e^{-1} and e . How does the accuracy compare to that of the previous two problems?

Note: The general formula for a cubic approximation of f centered at $x = 0$ (AKA the **3rd degree Taylor polynomial** of f centered at $x = 0$) is:

$$T(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3.$$

Check that your answer for $T(x)$ in part (b) matches this formula.

4. (a) Fill in the table below, for $f(x) = e^x$. Note that $f^{(n)}(x)$ means the n th derivative of $f(x)$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$	$\frac{f^{(n)}(0)}{n!} x^n$
0	e^x	1	$\frac{1}{0!} = 1$	1
1	e^x	1	$\frac{1}{1!} = 1$	x
2	e^x	1	$\frac{1}{2!} = \frac{1}{2}$	$\frac{1}{2}x^2$
3	e^x	1	$\frac{1}{3!} = \frac{1}{6}$	$\frac{1}{6}x^3$
4	e^x	1	$\frac{1}{4!} = \frac{1}{24}$	$\frac{1}{24}x^4$
5	e^x	1	$\frac{1}{5!} = \frac{1}{120}$	$\frac{1}{120}x^5$
6	e^x	1	$\frac{1}{6!} = \frac{1}{720}$	$\frac{1}{720}x^6$

- (b) Add the terms in the last column to get the 6th degree Taylor Polynomial for $f(x) = e^x$ centered at $x = 0$.

Solution: $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6$

- (c) Use the 6th degree Taylor polynomial to estimate e .

Solution: $e^1 \approx 1 + 1 + \frac{1}{2}(1)^2 + \frac{1}{6}(1)^3 + \frac{1}{24}(1)^4 + \frac{1}{120}(1)^5 + \frac{1}{720}(1)^6 = 2.7180\bar{5}$

5. Write down the general formula for the n th degree Taylor polynomial for a function $f(x)$, centered at $x = 0$.

Solution: $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$, or in summation notation $\sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k$

6. Give an explanation of where the factorials come from.

Solution: Each coefficient, C_k , in the Taylor polynomial is found by taking k derivatives of the polynomial, where k corresponds with the degree of that term in the polynomial. k derivatives means k successive powers of x have been brought down, which creates the factorial.