

6.6 - Center of Mass

First, computing total mass:



ex: a thin rod, 10 cm long

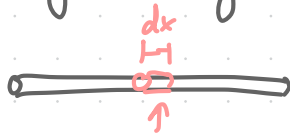
$$\rho = 9 \text{ g/cm} \quad (\text{linear mass density})$$



$$\text{mass} = \rho \cdot \text{length}$$

$$= 9 \cdot 10 = 90 \text{ g}$$

ex: density changes: $\rho(x) = 1 + \sqrt{x} \text{ g/cm}$



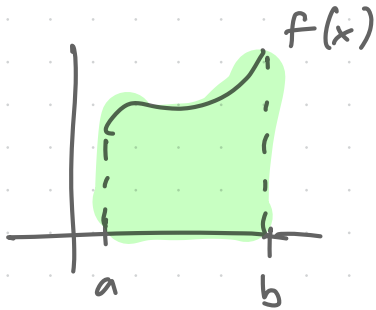
10 cm long



$$\begin{aligned} \text{mass of chunk} &= \rho(x) \cdot \text{length} \\ &= (1 + \sqrt{x})(dx) \end{aligned}$$

$$\text{total mass} = \int_0^{10} (1 + \sqrt{x}) dx$$

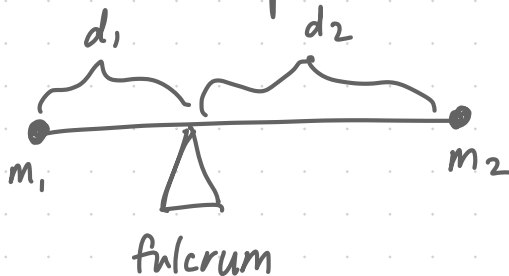
ex: mass of a 2d lamina (plate)



ρ is constant
 $\rho \text{ g/m}^2$

$$\begin{aligned}\text{mass} &= \rho \cdot \text{area of region} \\ &= \rho \cdot \int_a^b f(x) dx\end{aligned}$$

Start with a simple situation: a seesaw

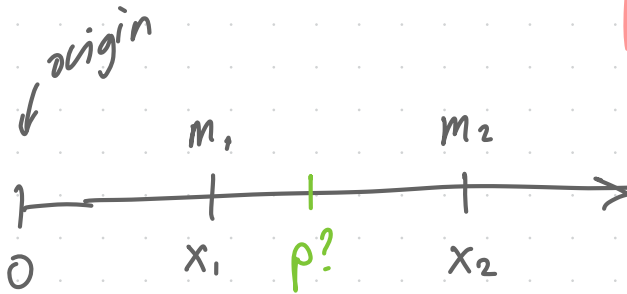


$$(m_1 > m_2) \quad \text{or} \quad m_1 < m_2$$

Lever Principle: For the seesaw to be balanced, $m_1 d_1 = m_2 d_2$.

$$(2m_2) d_1 = m_2 d_2$$

$$m_2 (2d_1) = m_2 \cdot d_2$$



What is p so that the seesaw is balanced?

Lever Principle: $m_1 (p - x_1) = m_2 (x_2 - p)$

∴ algebra

$$p = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

} sum of moments

$$\text{Center of Mass: } \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

total mass of the system

$m_1 x_1$ is the moment of the mass m_1 at location x_1 relative to the origin

" $m_1 x_1 + m_2 x_2$ " is the combined moment

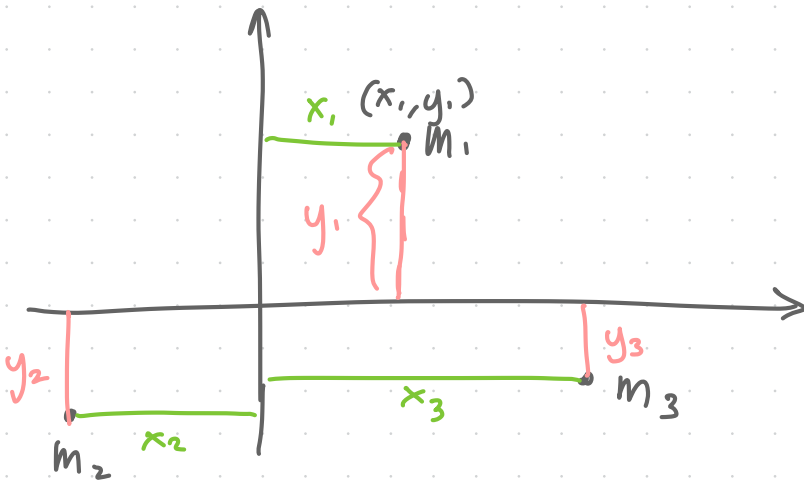
$$\bar{x} = \frac{\text{combined moment}}{\text{total mass}}$$

If we have a system of n point masses

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \dots + m_n x_n}{m_1 + m_2 + \dots + m_n}$$

$$= \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i}$$

Now consider point masses in xy-plane:



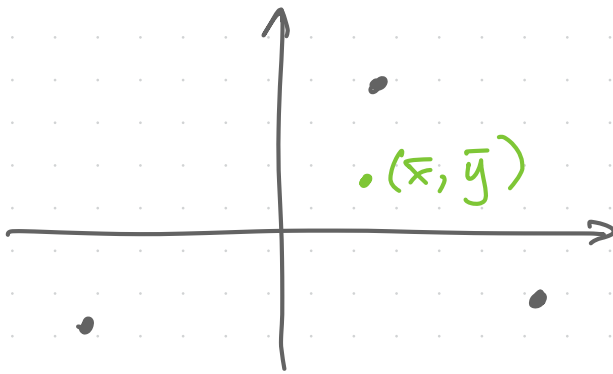
Two moments to consider: x-moment
and y-moment

For the x-moment, use the "distance"
to the x-axis: ↑ can be negative

$$M_x = m_1 y_1 + m_2 y_2 + m_3 y_3$$

For the y-moment, use dist. to y-axis:

$$M_y = m_1 x_1 + m_2 x_2 + m_3 x_3$$



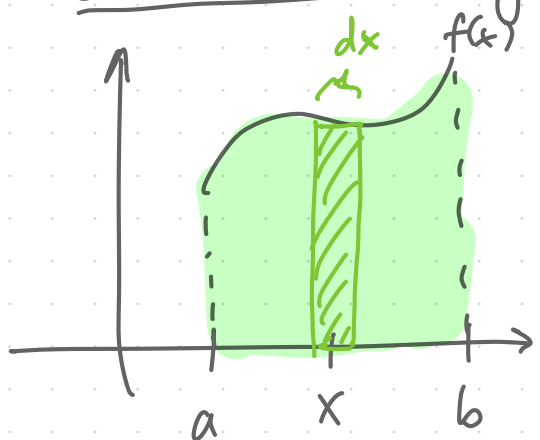
Center of Mass: (for system of 3 point masses in the plane)

$$\bar{x} = \frac{M_y}{m} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

↖ balance relative to y-axis for $\bar{x}$
↗ total mass

$$\bar{y} = \frac{M_x}{m} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3}$$

CoM of 2d region / continuum / lamina



Assume uniform density $\Rightarrow \rho$ is constant

$$\text{y-moment of a slice at } x = \text{mass} \cdot \left(\text{dist. to y-axis} \right)$$

$$= \rho \cdot (\text{area of slice}) \cdot x$$

$$= \rho \cdot (f(x) \cdot dx) \cdot x$$

$$= \rho x f(x) dx$$

$$M_y = \int_a^b \rho x f(x) dx$$

and total mass is

$$m = \rho \cdot \int_a^b f(x) dx$$

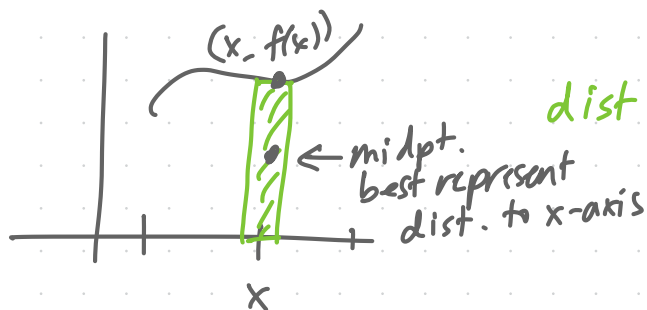
$$\bar{x} = \frac{M_y}{m} = \frac{\int_a^b \rho x f(x) dx}{\int_a^b \rho f(x) dx}$$

$$= \frac{\int_a^b x f(x) dx}{\int_a^b f(x) dx}$$

$$= \frac{1}{A} \int_a^b x f(x) dx$$

where A is the area under f on $[a, b]$

Now for $\bar{y}$, consider the x -moment



dist to x -axis = 0??

mid point of slice: $(x, \frac{1}{2} f(x))$

$$\begin{aligned}
 \text{x-moment of slice} &= (\text{mass of slice}) \cdot (\text{dist. to x-axis}) \\
 &= \rho \cdot (\text{area of slice}) \left(\frac{1}{2} f(x) \right) \\
 &= \rho (f(x) \cdot dx) \left(\frac{1}{2} f(x) \right) \\
 &= \frac{1}{2} \rho f(x)^2 dx
 \end{aligned}$$

$$M_x = \int_a^b \frac{1}{2} \rho f(x)^2 dx$$

$$\bar{y} = \frac{M_x}{m} = \frac{\int_a^b \frac{1}{2} \rho f(x)^2 dx}{\int_a^b \rho f(x) dx}$$

$$= \frac{\cancel{\rho} \int_a^b \frac{1}{2} f(x)^2 dx}{\cancel{\rho} \int_a^b f(x) dx}$$

$$= \frac{1}{A} \int_a^b \frac{1}{2} f(x)^2 dx$$

A is area under f

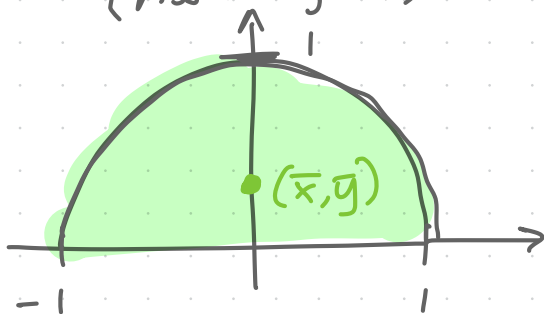
CoM Formulas

$$\bar{x} = \frac{1}{A} \int_a^b x f(x) dx$$

$$\bar{y} = \frac{1}{A} \int_a^b \frac{1}{2} f(x)^2 dx$$

← or CoM

ex: Find the centroid of the region.
(Assume $\rho=1$)



$$A = \frac{1}{2} (\pi (1)^2)$$

$$= \frac{\pi}{2}$$

top half of circle
 $y = \sqrt{1-x^2}$

$$\bar{x} = \frac{1}{(\pi/2)} \int_{-1}^1 x f(x) dx$$

$$= \frac{2}{\pi} \int_{-1}^1 x \sqrt{1-x^2} dx$$

$$= \frac{2}{\pi} \int_0^0 \sqrt{u} \left(-\frac{1}{2}\right) du$$

$$= 0$$

$$u = 1-x^2$$

$$du = -2x dx$$

$$-\frac{1}{2} du = x dx$$

$$x = -1 \Rightarrow u = 0$$

$$x = 1 \Rightarrow u = 0$$

$$\bar{y} = \frac{1}{A} \int_a^b \frac{1}{2} (f(x))^2 dx$$

$$= \frac{1}{(\pi/2)} \int_{-1}^1 \frac{1}{2} (\sqrt{1-x^2})^2 dx$$

$$= \frac{\cancel{2}}{\pi} \cdot \frac{\cancel{1}}{2} \int_{-1}^1 1-x^2 dx$$

$$= \frac{1}{\pi} \int_{-1}^1 1-x^2 dx$$

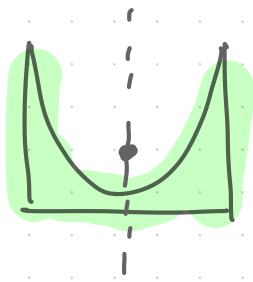
$$= \frac{1}{\pi} \left(x - \frac{1}{3} x^3 \right) \Big|_{-1}^1$$

$$= \frac{1}{\pi} \left(\left(1 - \frac{1}{3} \right) - \left(-1 - \frac{1}{3} (-1)^3 \right) \right)$$

$$= \frac{1}{\pi} \left(1 - \frac{1}{3} + 1 - \frac{1}{3} \right)$$

$$= \frac{1}{\pi} \left(\frac{4}{3} \right) = \frac{4}{3\pi} \approx 0.42$$

COM is $(\bar{x}, \bar{y}) = (0, 0.42)$



COM can lie outside
the region