

Turn in these problems:

means take the limit

1. Determine whether each sequence below converges or diverges. If the sequence converges, find its limit. Carefully show your work!

(a) $a_n = \sqrt[n]{7^{1+2n}}$

$= 7^{\frac{1+2n}{n}}$

(b) $b_n = \frac{(\ln n)^3}{n}$

L'Hospital's Rule

(c) $c_n = \frac{n!}{e^n}$

↑
n! grows faster,
"wins the race"

(d) $d_n = \left(1 + \frac{2}{n}\right)^{3n}$

log laws to bring exponent down

(e) $e_n = \frac{\sqrt{3+9n^4}}{3n^2+4}$

compare "degrees" or delete lower order terms

(f) $f_n = \frac{\sin 2n}{1+\sqrt{n}}$



Squeeze thm.

use: $-1 \leq \sin(2n) \leq 1$

2. Determine whether each series below is convergent or divergent. If the series is convergent, find its sum. Carefully show your work, and justify any tests that you use.

(a) $\sum_{k=1}^{\infty} \frac{(k+1)(3k-1)}{(k+3)^2}$

Div. Test.

(b) $\sum_{k=1}^{\infty} \ln\left(\frac{k}{k+1}\right)$

↑
 $\ln(k) - \ln(k+1)$

telescoping

(c) $\sum_{k=1}^{\infty} (\cos(5))^k$

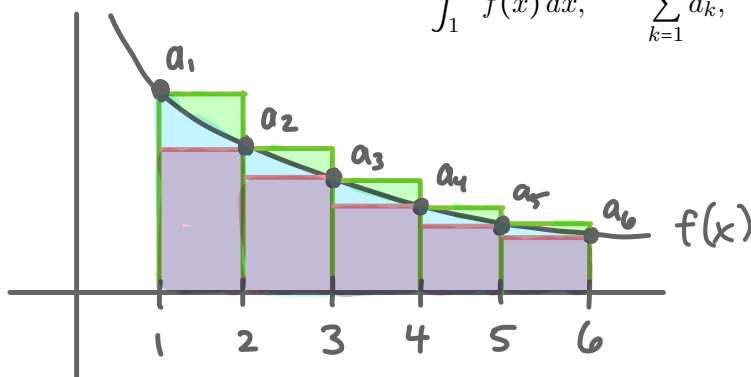
← geometric

(d) $\sum_{k=3}^{\infty} \frac{1}{k^2+k}$

← telescoping, PFD first (see example)

3. Suppose f is a continuous, positive, decreasing function for $x \geq 1$ and $a_k = f(k)$ for $k \geq 1$. By drawing a picture, rank the following three quantities in increasing order:

$$\int_1^6 f(x) dx, \quad \sum_{k=1}^5 a_k, \quad \sum_{k=2}^6 a_k.$$



find a formula for s_n (telescoping) OR geometric series have $\frac{a}{1-r}$

4. It is important to distinguish between

$$\sum_{k=1}^{\infty} k^b \quad \text{and} \quad \sum_{k=1}^{\infty} b^k.$$

geometric

- (a) What is the difference between these two series?
 (b) For what values of b does the second series converge?

*try
Div. Test*

5. Determine whether each series below is convergent or divergent. Justify any tests that you use.

(a) $\sum_{k=1}^{\infty} \frac{4 + 3^k}{2^k}$

(b) $\sum_{k=2}^{\infty} \frac{k}{\ln k}$

6. The following statements are **false**. For each statement, please provide a counterexample:

(a) If $\lim_{n \rightarrow \infty} a_n = 0$, then $\sum_n a_n$ converges.

(b) Let f be a continuous function and define a sequence $a_n = f(n)$. If $\lim_{n \rightarrow \infty} a_n$ exists and is equal to a finite real number L , then $\lim_{x \rightarrow \infty} f(x) = L$.

7. Explain why the Integral Test can or cannot be used to determine whether each series is convergent. You do not have to determine if the series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{\sqrt{n}}$

(b) $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{1 + n^2}$

(c) $\sum_{k=1}^{\infty} \frac{1}{(3k-1)^4}$

*Conditions out:
continuous, positive, decreasing*

8. The Reimann zeta-function ζ is defined by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

and is used in number theory to study the distribution of primes.

- (a) For values of x does ζ converge? (i.e. What is the domain of ζ ?) Explain your reasoning. $p > 1$
- (b) Leonhard Euler was able to calculate the exact sum of the p -series with $p = 2$ and $p = 4$

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Use Euler's results to find the sum of the following series:

I. $\sum_{n=3}^{\infty} \frac{1}{(n-2)^2}$ *reindex*

III. $\sum_{n=2}^{\infty} \frac{1}{n^2}$ *← subtract the 1st term*

II. $\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^4$ *←*

IV. $\sum_{n=1}^{\infty} \frac{1}{(n+3)^4}$ *← reindex, and subtract a few terms*

9. Recall from class this week:

The Integral Test:

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ where $a_n = f(n)$ for each $n \geq 1$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral

$$\int_1^{\infty} f(x) dx \text{ is convergent.}$$

- (a) State the hypotheses (or conditions) for the Integral Test.

- (b) State the conclusion for the Integral Test.

- (c) Give a brief explanation for why **each** hypothesis (or condition) is necessary for the Integral Test.

- (d) Explain what the phrase “if and only if” means in the Integral Test.

decreasing necessary for the geometric argument (see Int. Test notes)