Throughout the talk / $\mathbb{C}$

§ GIT

We begin with a variety X equipped with an action of an algebraic group G

A natural question is

> Is the orbit space X/G also a variety?

The answer is no in general: orbits may fail to be closed, stabilisers may vary wildly and separating orbits by regular functions can be impossible.

This motives the restriction to a special class of groups

① Def (Reductive group)

$G \subseteq GL_n(k)$ is reductive if it is a linear algebraic group with trivial unipotent radical

linear algebraic group: subgroup of $GL_n(k)$ defined by polynomials

$\Updownarrow$

$Ru(G)$ largest connected normal unipotent subgroup of G

connected: Topological sense

unipotent: Group theory sense

$(g - I_n)^m = 0$ for some

Intuitively, one can understand it as a group which only has "Pure symmetries" like rotations or scalings or orthogonal transformations. No shearing.

Ex: SL_n, $O_n = \{A \mid A^T A = I_n\}$ orthogonal gp

$SO_n = \{A \in O_n \mid \det(A) = 1\}$ special ..

$Sp_{2n}(k) := \{A \in GL_{2n}(k) \mid A^T J A = J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\}$

We classify $x \in X$ into the following types

Stable: Pts with closed $G \cdot x$ and finite G_x
They have well-behaved geometric quotients

Polystable: Pts with closed $G \cdot x$ but not necessarily of maximal dim.
They corresponds to an actual geometric pt of the quotient
(corresponds injectively to closed orbits.

semistable: Pts may degenerate under 1-PS.
but never degenerates to the worst possible orbit
(i.e. $y = \lim_{t \to 0} \lambda(t) \cdot x \notin$ Null cone
$= \{ v \in V \mid 0 \in \overline{G \cdot v} \}$
$G \curvearrowright X \subset \mathbb{P}(V)$ linearly

unstable: Pts can't be part of a well-defined quotient and are discarded in the construction.

② Now, If $X = \operatorname{Spec} A$ and G reductive, then A^G is finitely generated,

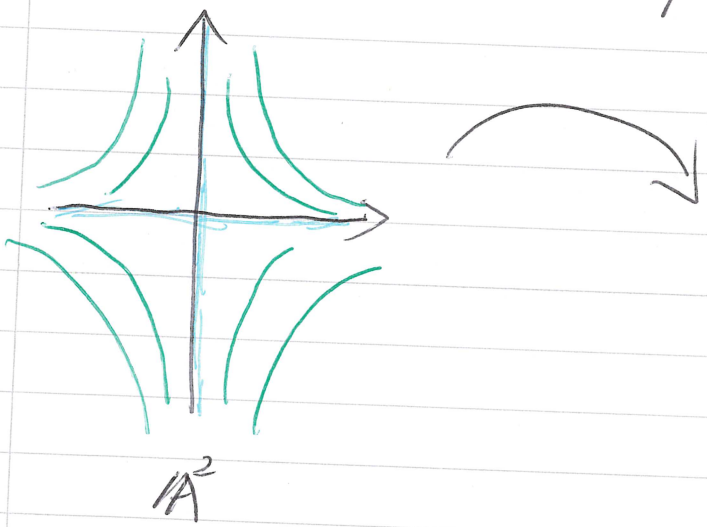
$\Rightarrow$ $X /\!\!/ G := \operatorname{Spec}(A^G)$

Ex: $\mathbb{C}^* \curvearrowright \mathbb{A}^2 = \operatorname{Spec}(\mathbb{C}[x,y])$ s.t.
$(x,y) \mapsto (tx, t^{-1}y)$, $t \in \mathbb{C}^*$

Then $\mathbb{C}[x,y]^{\mathbb{C}^*} = \mathbb{C}[xy]$

So $\mathbb{A}^2 /\!\!/ \mathbb{C}^* := \operatorname{Spec}(\mathbb{C}[xy]) = \mathbb{A}^1$

Get a morphism $\mathbb{A}^2 \longrightarrow \mathbb{A}^1$
$(x,y) \longmapsto xy$



$\mathbb{A}^2$

$\mathbb{A}^2 /\!\!/ \mathbb{C}^* = \mathbb{A}^1$

The pre-image of non-zero constant is closed.
(i.e. They all have closed orbits)
$\{(x,y) \mid xy \neq 0\}$
The pre-image of zero is open.
This leads to the stability notion of $x \in X$.

$x \in X$ stable if { 1) $G \cdot x$ closed
2) G_x has dim 0
(i.e. $\dim(G \cdot x) = \dim(G)$)
(i.e. G_x is a finite gp

$x \in X$ polystable if 1) satisfied.
In affine case, every pt is semi-stable

③ Generalize it to projective case.
need ample linearisation to make sense of H-M weight.

Def: An ample linearisation L is a G-equivariant ample line bundle on X

(So G acts compatibly on the total space of L covering its action on X)

Now G reductive $\curvearrowright X$ projective.
L ample linearisation

Let $R := \bigoplus_{r\geq 0} H^0(X, rL)$
f.g. k-algebra of global sections of powers of L.

$\Rightarrow R^G := \bigoplus_{r\geq 0} H^0(X, rL)^G$

$\Rightarrow X /\!\!/_L G := \mathrm{Proj}\left(\bigoplus_{r\geq 0} H^0(X, rL)^G = R^G\right)$

$X^{ss} \to X/\!\!/_L G$ good quotient. $X^s \to X/\!\!/_L G$ geometric quotient.

$x \in X$ semistable if $\exists\ \sigma \in H^0(X, rL)^G$ for some L s.t. $\sigma(x) \neq 0$

$x \in X$ stable if (1) $x \in X^{ss}$ $\Leftrightarrow$ morphism $X_\sigma \to \mathbb{P}(H^0(X, rL)^G)$ separates orbits near x
(2) $G \cdot X_\sigma$ is closed in X_σ
where X_σ non-vanishing locus of σ
$X_\sigma := \{x \in X \mid \sigma(x) \neq 0\}$.
(3) G_x finite

$x \in X$ polystable, if (1) $x \in X^{ss}$
(2) $G \cdot x$ is closed in X^{ss}

④ In practice, we check the stability of a p by looking at its H-M weight.

- Given a 1-PS $\lambda : \mathbb{G}_m \to G$ (i.e. a non-trivial gp homomorphism)
- Fix $x \in X$, define $\lambda_x : \mathbb{G}_m \to X$
  $t \mapsto \lambda(t) \cdot x$
- Let $y := \lim_{t \to 0} \lambda(t) \cdot x$ be the limit pt
  Then the H-M weight $\mu(x, \lambda)$ w.r.t. λ is the weight of λ on λy

$$\mu(x, \lambda) := \min \{ r_i \mid x_i \neq 0 \}.$$

where $\lambda(t) \cdot \tilde{x} = \sum_{i=0}^{n} t^{r_i} x_i e_i$ ← basis of $\mathbb{A}^{n+1}$

$\tilde{x}$: lift of x in affine cone.

- H-M criterion:
  $x \in X^{ss} \iff \mu(x, \lambda) \leq 0$, $\forall$ 1-PS λ of G.
  $x \in X^{s} \iff \mu(x, \lambda) < 0$, $\forall$ 1-PS λ of G.

§ K-stability

Now we have the stability notion of a pt ~~Test~~ $x \in X$. Can we have a stability notion of a variety ? That motivates K-stability.

① Test configuration $(\mathcal{X}, \mathcal{L})$ of (X, L)

1) $\mathcal{X} \to \mathbb{C}$ is a flat morphism of schemes
(s.t. Hilb poly of $(\mathcal{X}_t, \mathcal{L}_t)$ independent of t)

2) $\mathcal{L}$ relatively ample $\mathbb{C}^*$-equivariant line bundle. ($f: L \to X$ s.t. $f(t \cdot v) = t \cdot f(v)$)

3) $\mathbb{C}^*$-action on $\mathcal{X}$ covering the standard $\mathbb{C}^*$-action on $\mathbb{C}$

4) Generic fiber $(\mathcal{X}_t, \mathcal{L}_t) \cong (X, rL)$ for some r, $t \neq 0$.
r: exponent of $(\mathcal{X}, \mathcal{L})$

So we degenerate (X, L) to $(\mathcal{X}_0, \mathcal{L}_0)$ in a $\mathbb{C}^*$-equivariant way

② $DF(\mathcal{X}, \mathcal{L})$

Given $\dim(X) = n$.

Let $V_r := H^0(\mathcal{X}_0, r\mathcal{L}_0)$

Let $wt(r) :=$ weight of $\mathbb{C}^*$-action on $H^0(\mathcal{X}_0, r\mathcal{L}_0) = b_0 r^{n+1} + b_1 r^n + \cdots$

Let $h(r) := \dim(H^0(\mathcal{X}_0, r\mathcal{L}_0)) = a_0 r^n + a_1 r^{n-1} + \cdots$

$$F(r) := \frac{wt(r)}{r \cdot h(r)} = F_0 + F_1 k^{-1} + F_2 k^{-2} + \cdots$$

!!

$$DF(\mathcal{X}, \mathcal{L}) = \frac{b_0 a_1 - b_1 a_0}{a_0^2}$$

③

(X, L) K-semistable iff $\forall$ non-trivial i.e. $\mathcal{X}_0 \neq X$
$(\mathcal{X}, \mathcal{L})$,
$DF(\mathcal{X}, \mathcal{L}) \leq 0$

" K-stable " $DF(\mathcal{X}, \mathcal{L}) < 0$

" K-polystable " $DF(\mathcal{X}, \mathcal{L}) \leq 0$
and $DF(\mathcal{X}, \mathcal{L}) = 0$
$\Leftrightarrow$ ~~$\cong$~~ $(\mathcal{X}, \mathcal{L})$
$= (X \times \mathbb{C}, \pi_1^* L)$

§ Bridge

① • Given $\mathbb{C}^*$-action on (X, L)

⇓

• $\mathbb{C}^*$-action on $H^0(X_0, rL_0)$

Pb: $\mathbb{C}^*$-action may not be SL, and we want it to be SL, because we don't want the DF-invariant to detect trivial scalings of the polarization

⇓

a Compose it with another action that scales the fiber L_0

$$t \longmapsto t^{\alpha}$$

Then the weight of the new $\mathbb{C}^*$-action becomes

$$w(r) \longmapsto w(r) + \alpha r h(r)$$

and we require $w(r) + \alpha r h(r) = 0$

② Normalised weight and Hilbert stability:

Consider polarised variety (X, L)

Let I_X be the ideal sheaf of X

Then s.e.s.

$$0 \to I_X \to \mathcal{O}_{\mathbb{P}^N}(1) \xrightarrow{= rL} \mathcal{O}_X(1) \to 0$$

($\mathcal{O}_{\mathbb{P}^N}(1) = rL_0$)

⇓ $\otimes\, \mathcal{O}_{\mathbb{P}^N}(m)$ ($\mathcal{O}_{\mathbb{P}^N}(m) = rmL_0$)

$$0 \to I_X \otimes \mathcal{O}_{\mathbb{P}^N}(m) := I_X(m) \to \mathcal{O}_{\mathbb{P}^N}(m) \to \mathcal{O}_X(m) \to 0$$

⇓ Taking cohomology

$$0 \to H^0(I_X(m)) \to S^m H^0(\mathcal{O}_{\mathbb{P}^N}(1)) \to H^0(\mathcal{O}_X(m)) \to H^1(I_X(m)) \to \cdots$$

Choose m s.t. the higher cohomology vanishes

⇓

$$0 \to H^0(I_X(m)) \to S^m H^0(\mathcal{O}_{\mathbb{P}^N}(1)) \to H^0(\mathcal{O}_X(m)) \to 0$$

($S^m H^0(\mathcal{O}_{\mathbb{P}^N}(1)) = rmL_0$)

Pts corresponding to subsche of $\mathbb{P}^{h(r)-1}$ with Hilb poly $h'(k) = h(rk)$

Alternating sum of chin cohomology gps o

- The image of this morphism gives us a pt $x_{r,k}$ in the Grassmannian
$$\mathrm{Grass}(S^m H^0(\mathcal{O}_{\mathbb{P}^N}(1)),\ h(rm))$$

- $x_{r,k}$ is the pt in the ~~Hilb~~ $\mathrm{Hilb}^{h,k}(\mathbb{P}^{h(r)-1})$ that corresponds to ~~X~~ X $(\mathcal{X}_0)$

- (X, L) is Hilbert (semi) stable if $x_{r,k}$ is .. in GIT sense for $k \gg 0$.

- The tautological line bundle over $x_{r,k}$ is
$$\Lambda = \Lambda^{\max} H^0(\mathcal{I}_{\mathcal{X}_0}(m))$$
$$= (\Lambda^{\max} H^0(\mathcal{O}_{\mathcal{X}_0}(m)))^* \otimes \Lambda^{\max} S^m H^0(\mathcal{O}_{\mathbb{P}^N}(1))$$

* s.e.s $\Rightarrow S^m H^0(\mathcal{O}_{\mathbb{P}^N}(1)) \cong H^0(\mathcal{I}_{\mathcal{X}_0}(m)) \oplus H^0(\mathcal{O}_{\mathcal{X}}(m))$

$\Rightarrow \Lambda^{\max}(\ldots) = \Lambda^{\max}(\ldots) \otimes \Lambda^{\max}(\ldots)$

A B C

$$wt(A) = wt(B) + wt(C)$$
$$wt(B) = wt(A) - wt(C)$$
$$= wt(A) + wt(C^*)$$

- The normalised weight $\tilde{w}(r,k)$ is the H-M weight of the $\mathbb{C}^*$-action on $\Lambda_{x_{r,k}}$

③ Thm: $DF(\mathcal{X}, \mathcal{L})$ has the same sign as the leading order term of $\tilde{w}(r,k)$

§ Generalise to log pair (X, D)

① log t.c. $(\mathcal{X}, \mathcal{D}, \mathcal{L})$ of (X, D, L):
- $(\mathcal{X}, \mathcal{L})$ t.c. of (X, L)
- $(\mathcal{D}, \mathcal{L}_{\mathcal{D}})$ t.c. of $(D, L|_D)$

s.t. they have the same exponent inside $(\mathcal{X}, \mathcal{L})$ and a compatible $\mathbb{C}^*$-action

$\mathcal{D} := \overline{\mathbb{C}^*(D \times (\mathbb{A}^1 \setminus \{0\}))}$

② log DF $(\mathcal{X}, \mathcal{D}, \mathcal{L})$ with cone angle $2\pi\beta$:
- $\dim H^0(\mathcal{X}_0, m\mathcal{L}_0) = a_0 m^n + a_1 m^{n-1} + \cdots$
- $W_{\mathcal{X}_0}(m) = b_0 m^{n+1} + b_1 m^n + \cdots$
- $\dim H^0(\mathcal{D}_0, m\mathcal{L}_0|_{\mathcal{D}_0}) = \tilde{a}_0 m^{n-1} + \tilde{a}_1 m^{n-2} + \cdots$
- $W_{\mathcal{D}_0}(m) = \tilde{b}_0 m^n + \tilde{b}_1 m^{n-1} + \cdots$

$$DF_\beta(\mathcal{X}, \mathcal{D}, \mathcal{L}) = DF(\mathcal{X}, \mathcal{L}) + (1-\beta)(a_0\tilde{b}_0 - b_0\tilde{a}_0)$$

of $DF(\mathcal{X}, \mathcal{L})$

$= 2(b_0 a_1 - b_1 a_0)$

If β is changed, the log K-stability may change as well.

③ log GIT
- The ample linearisation:

$$\mathcal{X}_0 \hookrightarrow \mathbb{P}(H^0(\mathcal{X}_0, k\mathcal{L}_0)^*)$$
$$\mathcal{D}_0 \hookrightarrow \mathbb{P}(H^0(\mathcal{D}_0, k\mathcal{L}_0|_{\mathcal{D}_0})^*)$$

~~$\mathcal{X}_{r,k}$~~

- The Hilbert pt $[\mathcal{X}_0] =: \mathcal{X}_{r,k}$ $\longleftrightarrow$ $([\mathcal{X}_0], [\mathcal{D}_0])$
- The tautological line bd $\Lambda_{\mathcal{X}_0} = \Lambda^{\max} H_0(\mathcal{I}_{\mathcal{X}_0}(m))$ $\longleftrightarrow$ $\Lambda_{\mathcal{X}_0} \otimes (\Lambda_{\mathcal{D}_0}^{\otimes\beta})$
- Normalised weight $\tilde{w}(k, m)$ $\longleftrightarrow$ $\tilde{w}_{\mathcal{X}_0}(k, m) + \beta \tilde{w}_{\mathcal{D}_0}(k, m)$