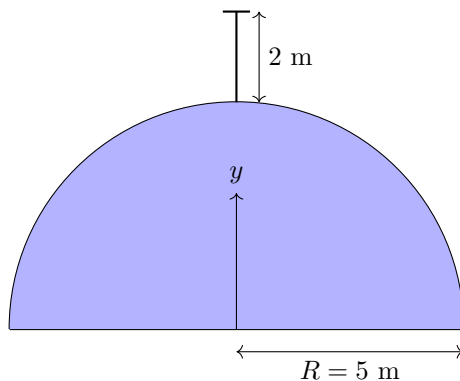
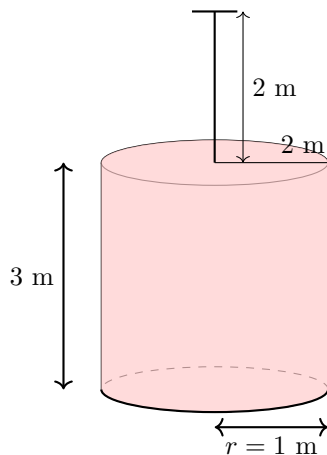


Exam 2- Practice**(i) Work**

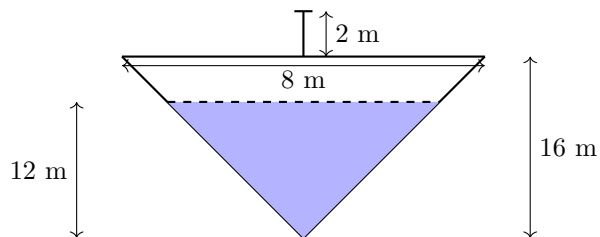
- (1) A hemispherical tank of radius 5 meters is full of water. The tank is positioned with its flat face at the bottom. Find the work required to pump all the water out through a pipe that extends 2 meters above the top of the tank. (Use $\rho = 1000 \text{ kg/m}^3$ for water density)



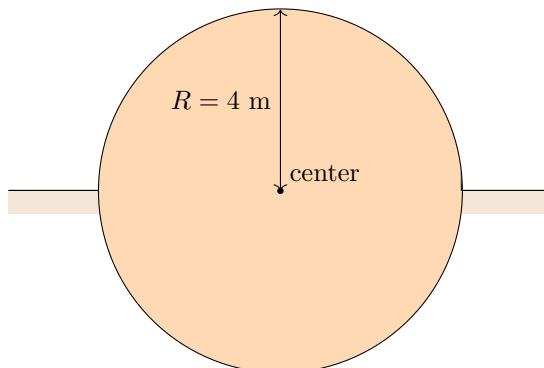
- (2) A vertical cylindrical tank with radius 1 meters and height 3 meters contains a liquid whose density varies with depth according to $\rho(y) = 8 + 5y$ kg/m³, where y is measured from the bottom. Find the work needed to pump all the liquid to a level 2 meters above the top of the tank.



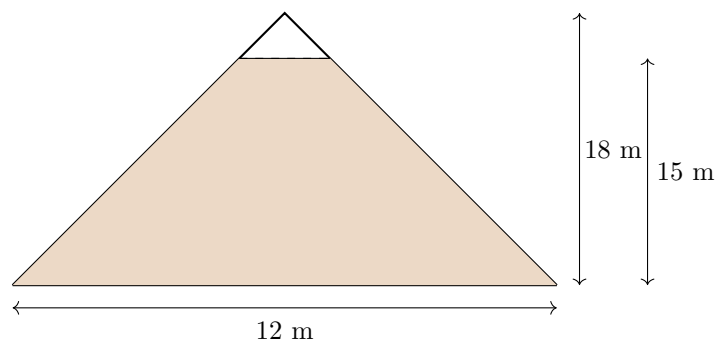
- (3) Consider an inverted pyramidal tank containing liquid with density ρ kg/m³. The tank has a square cross-section with 8-meter sides at the top and comes to a point at the bottom, with a total height of 16 meters. If the tank currently contains liquid up to a depth of 12 meters, set up an integral expression that would calculate the work needed to pump all the liquid out through an outlet located 2 meters above the top of the tank. Do not evaluate the integral.



- (4) A spherical tank of radius 4 meters is buried in the ground with its center at ground level. The tank is full of gasoline ($\rho = 720 \text{ kg/m}^3$). Find the work required to pump all the gasoline to a height of 5 meters above ground level.



- (5) A pyramidal storage tank sits on its base and tapers to a point at the top. The base is a square measuring 12 meters on each side, and the tank reaches a height of 18 meters. The tank holds liquid (with density ρ kg/m³) filled to a level of 15 meters. Express as an integral the total work that would be required to remove all the liquid by pumping it out at the top of the tank.



(ii) **Center of mass**

(1) Find the centroid of the region bounded by $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \frac{\pi}{4}$.

(2) Find the arc length of the curve $y = \frac{1}{3}(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$.

(3) Find the arc length of the curve $y = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2}$ from $x = 1$ to $x = 9$.

(4) Find the arc length of the curve $y = \frac{2}{3}(x - 1)^{3/2}$ from $x = 1$ to $x = 4$.

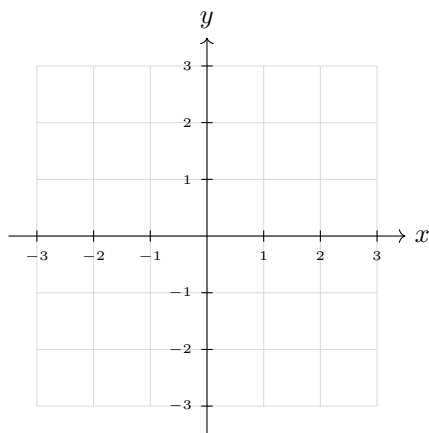
(5) Find the arc length of the curve $y = \frac{1}{2}(e^x + e^{-x})$ from $x = 0$ to $x = 2$.

(6) Find the centroid of the region bounded by $y = 4 - x^2$ and the x -axis.

(iii) **Parametric equations**

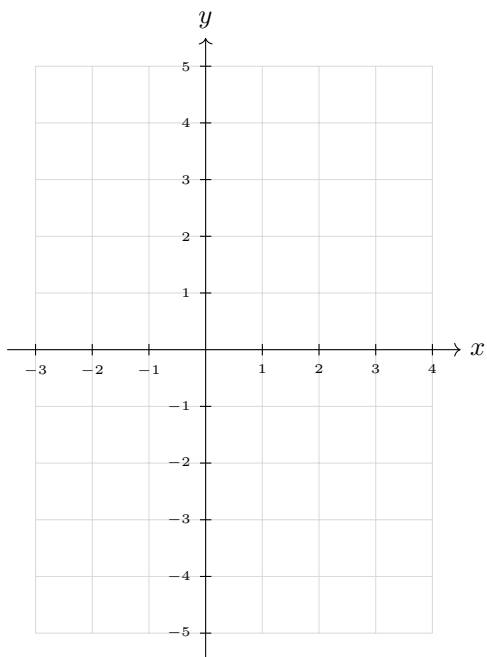
(1) Draw the parametrized curve given by

$$\begin{cases} x = 2 \sin t \\ y = 2 \cos t \end{cases} \quad 0 \leq t \leq \pi.$$



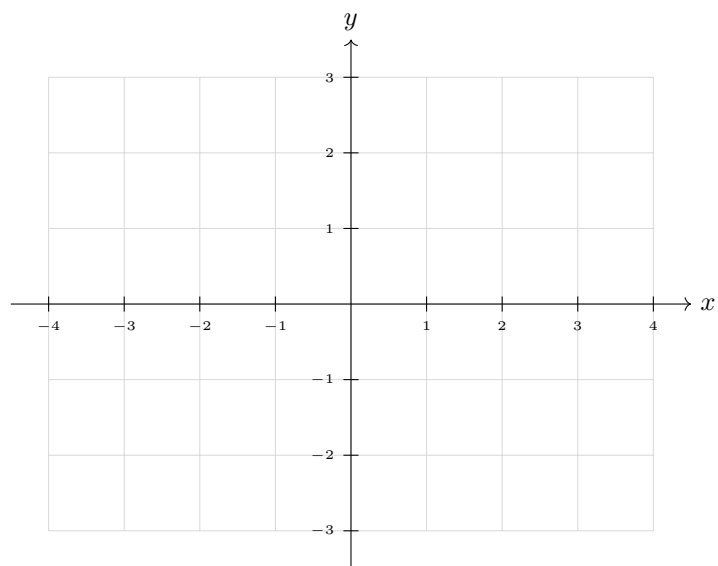
(2) Draw the parametrized curve given by

$$\begin{cases} x = t^2 - 1 \\ y = 2t \end{cases} \quad -2 \leq t \leq 2.$$



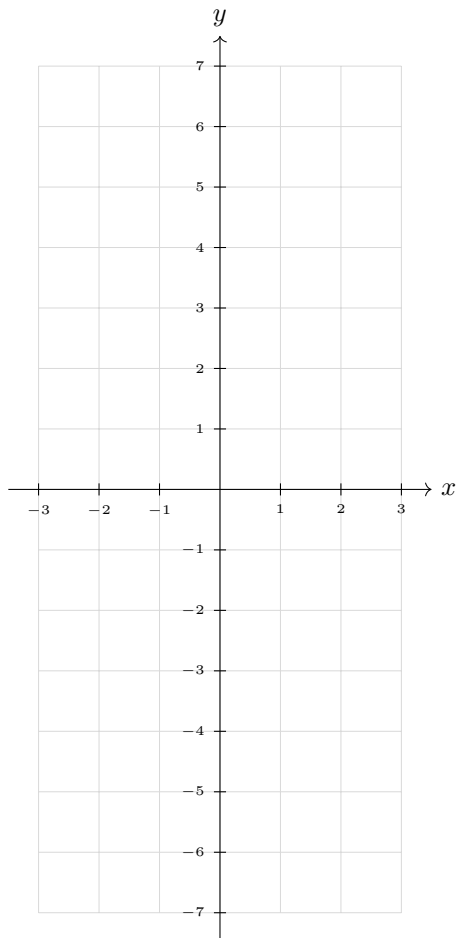
(3) Draw the parametrized curve given by

$$\begin{cases} x = 3 \cos t \\ y = 2 \sin t \end{cases} \quad 0 \leq t \leq 2\pi.$$



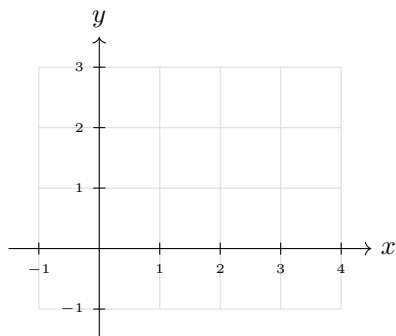
(4) Draw the parametrized curve given by

$$\begin{cases} x = t \\ y = t^3 - t \end{cases} \quad -2 \leq t \leq 2.$$



(5) Draw the parametrized curve given by

$$\begin{cases} x = 2 + \cos t \\ y = 1 + \sin t \end{cases} \quad 0 \leq t \leq 2\pi.$$



(iv) **Tangents of parametric Curves**

(1) Consider the parametric equations:

$$x = t^2 - 4$$

$$y = t^3 - 3t$$

- a) Find $\frac{dy}{dx}$ in terms of t .
- b) Find the equation of the tangent line at $t = 2$.
- c) Find all points where the tangent line has slope 0.

(2) For the curve defined by:

$$x = 3 \cos(t)$$

$$y = 2 \sin(t)$$

$$0 \leq t \leq 2\pi$$

- a) Find all points where the curve has a horizontal tangent line.
- b) Find all points where the curve has a vertical tangent line.
- c) Write the equation of the tangent line at $t = \pi/4$.

(3) For the curve defined by:

$$x = t^3 + 4t + 5$$

$$y = t^2 + 6t$$

$$0 \leq t$$

- a) Find all points where the curve has a horizontal tangent line (If they exist).
- b) Find all points where the curve has a vertical tangent line (If they exist).
- c) Write the equation of the tangent line at $t = 0$.

(4) The curve defined by:

$$x = t^2 - 1$$

$$y = t^3 - t$$

crosses itself at the point $(0, 0)$.

- a) Find the two values of t that give the point $(0, 0)$.
- b) Find the slopes of both tangent lines at this point.

(5) For the logarithmic spiral:

$$x = e^t \cos(t)$$

$$y = e^t \sin(t)$$

$$0 \leq t \leq 2\pi$$

a) Show that $\frac{dy}{dx} = \frac{\sin(t) + \cos(t)}{\cos(t) - \sin(t)}$.

b) Find all values of t where the tangent line passes through the origin.

(v) **Arc Length**

(1) Write the integral that represents the arc length of the curve $y = x^2$ from $x = 0$ to $x = 2$.

(2) Write the integral that represents the arc length of the curve $y = \ln(x)$ from $x = 1$ to $x = e$.

- (3) Write the integral that represents the arc length of the curve $y = (2x - 4)^{5/2}$ from $x = 1$ to $x = 5$.

- (4) Write the integral that represents the arc length of the parametric curve

$$\begin{cases} x = t^3 \\ y = t^2 \end{cases}$$

from $t = 0$ to $t = 2$.

- (5) Write the integral that represents the arc length of the curve $x = \sqrt{y}$ from $y = 1$ to $y = 4$.

(vi) **Polar Coordinates**

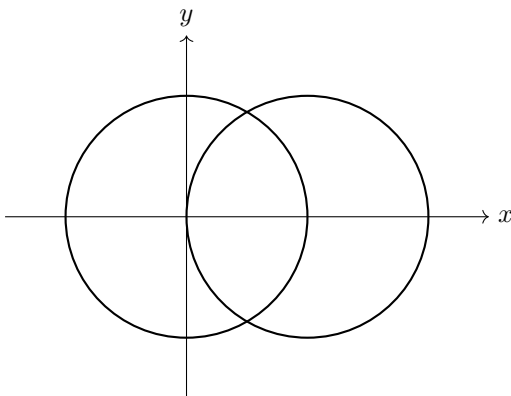
- (1) Convert the point with polar coordinates $(r, \theta) = (4, \frac{\pi}{3})$ to Cartesian coordinates.
- (2) Convert the point with polar coordinates $(r, \theta) = (3, \frac{3\pi}{4})$ to Cartesian coordinates.
- (3) Convert the point with polar coordinates $(r, \theta) = (5, -\frac{\pi}{6})$ to Cartesian coordinates.
- (4) Convert the point with Cartesian coordinates $(x, y) = (3, 3)$ to polar coordinates.

(5) Convert the point with Cartesian coordinates $(x, y) = (0, -5)$ to polar coordinates.

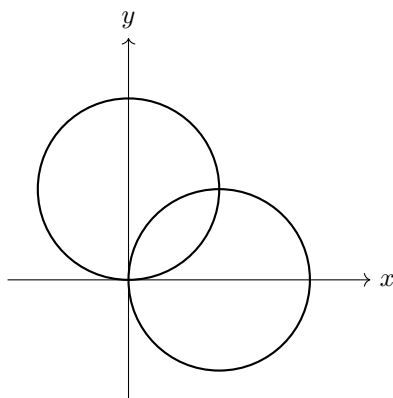
(6) Convert the point with Cartesian coordinates $(x, y) = (-4, -4)$ to polar coordinates.

(vii) **Areas in polar coordinates**

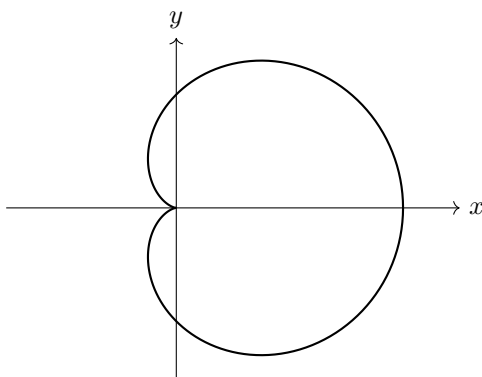
(1) Write an integral to compute the area of the region inside $r = 4 \cos \theta$ and outside $r = 2$.



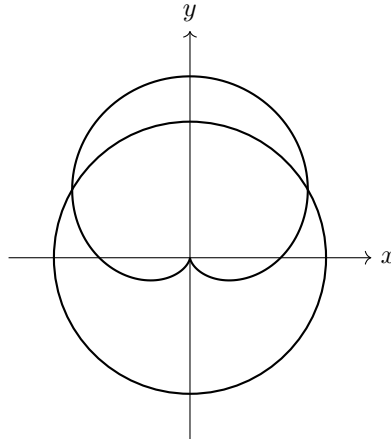
- (2) Write an integral to compute the area of the region inside both $r = 3 \sin \theta$ and $r = 3 \cos \theta$.



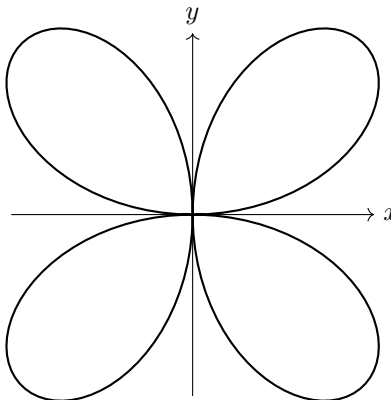
- (3) Write an integral to compute the area of the region inside $r = 1 + \cos \theta$ (cardioid).



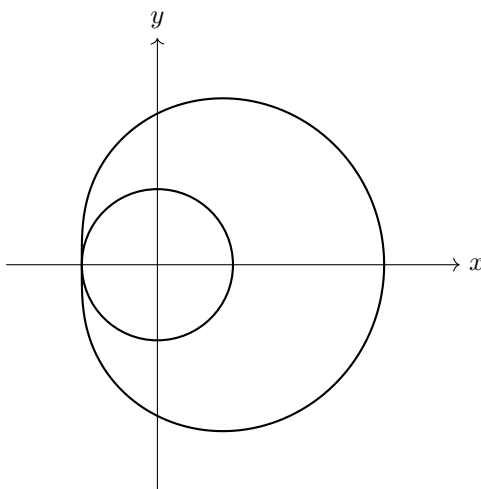
- (4) Write an integral to compute the area of the region inside $r = 2 + 2 \sin \theta$ and outside $r = 3$.



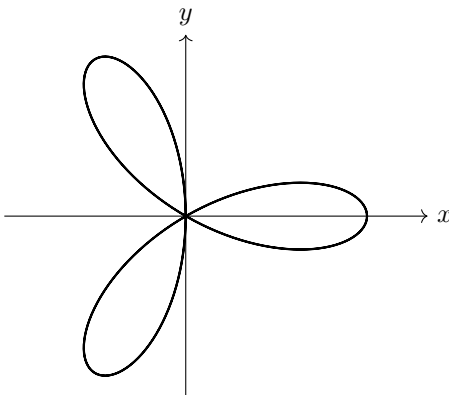
- (5) Write an integral to compute the area of the region inside $r = 4 \sin(2\theta)$ (four-petaled rose).



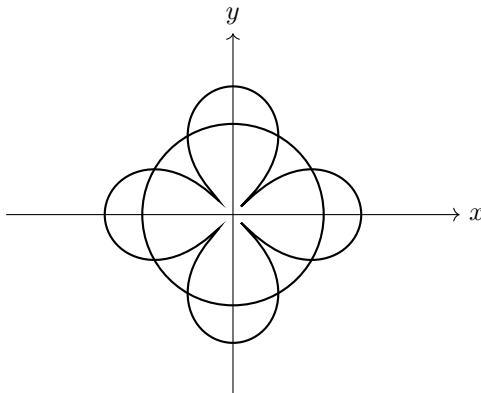
- (6) Write an integral to compute the area of the region inside $r = 2 + \cos \theta$ and outside $r = 1$.



- (7) Write an integral to compute the area of the region inside $r = 3 \cos(3\theta)$ (three-petaled rose).



- (8) Write an integral to compute the area of the region inside $r = 2 \cos(2\theta)$ (lemniscate) and outside $r = 1$.



(viii) **Arc Length in polar coordinates**

- (1) Find the arc length of the circle $r = 4$ from $\theta = 0$ to $\theta = \pi/2$.

- (2) Find the total arc length of the cardioid $r = 2(1 + \cos \theta)$ for $\theta \in [0, 2\pi]$.

- (3) Calculate the arc length of the spiral $r = \theta$ from $\theta = 0$ to $\theta = 2\pi$.
- (4) Find the arc length of one petal of the four-petaled rose $r = 3 \sin(2\theta)$ from $\theta = 0$ to $\theta = \pi/2$.
- (5) Determine the arc length of the logarithmic spiral $r = e^{\theta/2}$ from $\theta = 0$ to $\theta = \pi$.

(ix) **Differential equations**

- (1) Check that $y = x^2 + \frac{1}{x}$ is a solution to the differential equation:

$$xy' + y = 3x^2$$

Also verify that it satisfies the initial condition $y(1) = 2$.

- (2) Show that $y = \tan(2x^2)$ is a solution to the differential equation:

$$\frac{dy}{dx} = 4x(1 + y^2)$$

and verify that it satisfies the initial condition $y(0) = 0$.

- (3) Verify that $x = t^2 - 2t + 3$ is a solution to the initial value problem:

$$\begin{aligned}\frac{d^2x}{dt^2} - 2 &= 0 \\ x(0) &= 3 \\ x'(0) &= -2\end{aligned}$$

- (4) Solve the differential equation:

$$\frac{dy}{dx} = \frac{y^2}{x^2}$$

with the initial condition $y(1) = 2$.

- (5) Find the general solution to:

$$(1 + e^x)y \, dy = e^x \, dx$$

- (6) The population of a small industrial city has been declining since a major factory closed. The population follows the model $\frac{dP}{dt} = -0.025P$, where P is measured in thousands of people and t is in years. In the year 2020, the population was 48,000.
- (a) Find the population function $P(t)$ where $t = 0$ corresponds to the year 2020.
 - (b) What will be the population in 2030?
 - (c) In what year will the population drop below 30,000?
 - (d) What is the half-life of the population (time for the population to decrease by half)?

(x) **Sequences**

(1) Calculate the following limit:

$$\lim_{n \rightarrow \infty} (-1)^n \frac{1}{n^2}$$

Does the sequence converge or diverge?

(2) Calculate the following limit:

$$\lim_{n \rightarrow \infty} \sin\left(\frac{1}{n}\right)$$

Does the sequence converge or diverge?

(3) Evaluate:

$$\lim_{n \rightarrow \infty} \frac{n^2 + 3n - 1}{2n^2 - n + 5}$$

Does the sequence converge or diverge?

(4) Find the limit:

$$\lim_{n \rightarrow \infty} \frac{e^n}{n}$$

Does the sequence converge or diverge?

(5) Determine whether the following sequence converges, and if so, find its limit:

$$a_n = \frac{n+2}{\ln(n+5)}$$

Does the sequence converge or diverge?

(xi) **Series**

Determine whether or not the series below converges or diverges. If it converges, determine to what value.

$$(1) \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+2} \right)$$

$$(2) \sum_{n=2}^{\infty} \ln \left(\frac{n-1}{n+1} \right)$$

$$(3) \sum_{k=0}^{\infty} \left(\frac{2}{3} \right)^k$$

$$(4) \sum_{n=1}^{\infty} \frac{n^2}{2n^2 + 5}$$

$$(5) \sum_{n=1}^{\infty} \frac{3n}{n+100}$$

$$(6) \sum_{n=1}^{\infty} (-1)^n$$

$$(7) \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{2n+1}\right)$$

$$(8) \sum_{n=1}^{\infty} \frac{2}{2^n}$$

$$(9) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

$$(10) \sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n}$$

$$(11) \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$(12) \sum_{n=1}^{\infty} \frac{2n+3}{5n+1}$$

$$(13) \sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$