1. For each of the following situations, determine if $\sum_{n=1}^{\infty} c_n$ converges, diverges, or if one cannot tell without more information.

   (a) If $0 \leq c_n \leq \frac{1}{n^2}$ for all $n$, we can conclude that $\sum c_n$ ________________

   (b) If $\frac{1}{n^2} \leq c_n$ for all $n$, we can conclude that $\sum c_n$ ________________

   (c) If $0 \leq c_n \leq \frac{1}{n}$ for all $n$, we can conclude that $\sum c_n$ ________________

   (d) If $\frac{1}{n} \leq c_n$ for all $n$, we can conclude that $\sum c_n$ ________________

   (e) If $\frac{1}{n^2} \leq c_n \leq \frac{1}{n}$ for all $n$, we can conclude that $\sum c_n$ ________________

2. **Follow-up to problem 1:** For each of the cases above where you cannot determine convergence/divergence, give one example of a series that satisfies the given conditions and **converges** and one example of a series that satisfies the given conditions and **diverges**.

3. Fill in the blanks:

   **The Comparison Test** (also known as Term-size Comparison Test or Direct Comparison Test)
   Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

   - If $\sum b_n$ ________________ and $a_n \leq b_n$, then $\sum a_n$ also ________________.

   - If $\sum b_n$ ________________ and $a_n \geq b_n$, then $\sum a_n$ also ________________.

   Note: In the statement of this test and for the rest of this worksheet...

   - use $\sum a_n$ to represent the given series (the one we are trying to determine convergence/divergence of)
   - use $\sum b_n$ represent the series whose convergence/divergence we already know (often a $p$-series or geometric series).

Now we'll practice using the Comparison Test.

4. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + n}$.

   - Let $a_n = \dfrac{1}{2^n + n}$ and choose $b_n = \dfrac{1}{2^n}$ for $n \geq 1$. (Notice that both $a_n$ and $b_n$ are sequences with positive terms.)

   - Does the series $\displaystyle\sum_{n=1}^{\infty} b_n$ converge or diverge? How do you know?

   - How do the size of the terms $a_n$ and $b_n$ compare?

   - What can you conclude about $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + n}$?

5. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + n + 1}$.

   - Let $a_n = \dfrac{1}{n^2 + n + 1}$ and choose $b_n =$ ____________. (Hint: consider the rate of growth of the denominator.)

   - Check that both $a_n$ and $b_n$ are sequences with positive terms.

   - Does the series $\displaystyle\sum_{n=1}^{\infty} b_n$ converge or diverge? How do you know?

   - How do the size of the terms $a_n$ and $b_n$ compare?

   - What can you conclude about $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + n + 1}$?

6. Use the Comparison Test to determine if $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{\sqrt{n^3+n}}$ converges or diverges.

7. Use the Comparison Test to determine if $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n^4+1}}{n^3-2}$ converges or diverges.

8. Disappointingly, sometimes the Comparison Test doesn't work like we want it to. For example, consider the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$.

   (a) A natural choice for $b_n$ is $b_n = \dfrac{1}{n^2}$ for $n \geq 2$. Does $\displaystyle\sum_{n=2}^{\infty} b_n$ converge or diverge?

   (b) How do the size of the terms $a_n$ and $b_n$ compare?

   (c) Using the Comparison Test, can you make any conclusion about $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$?

   (d) Show that $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

   (e) Since $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, we know that $a_n \approx b_n$ for large values of $n$. Justify why this means that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ will converge.

When we have chosen a good series to compare to but the inequalities don't work in our favor, we can use the Limit Comparison Test instead of the Comparison Test.

**The Limit Comparison Test**

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c, \text{ where } c > 0 \text{ and } c \text{ is finite}$$

then either both series ______________ or both series _____________.

Now we'll practice using the Limit Comparison Test.

9. Consider the series $\displaystyle\sum_{n=2}^{\infty} \frac{n^3 - 2n}{n^4 + 3}$.

   (a) Let $a_n = \dfrac{n^3 - 2n}{n^4 + 3}$ and choose $b_n =$ ________________.

   (b) Does $\displaystyle\sum_{n=2}^{\infty} b_n$ converge or diverge?

   (c) Compute $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n}$.

   (d) Is $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n}$ positive and finite? If yes, make a conclusion using the Limit Comparison Test.