

Late Time Virasoro Blocks and the Information Paradox

Thesis: On the quantum nature of the horizon

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Structure of the presentation

- Introduction and setting up the stage
- Progress so far
- Goals and results we can expect going forward

Introduction

The Black Hole Information Paradox

Since Hawking's seminal paper in 1975, the study of black hole thermodynamics has been topic of immense scrutiny and deliberation. This is because Hawking's calculation seemed to conflict with one of the basic postulates of quantum mechanics: the unitarity of time evolution. The apparent contradiction between Hawking's calculation and the unitarity postulate of quantum mechanics came to be known as the "black hole information paradox". More generally, the information loss problem is the inconsistency of the equivalence principle against the unitary quantum mechanical evolution.

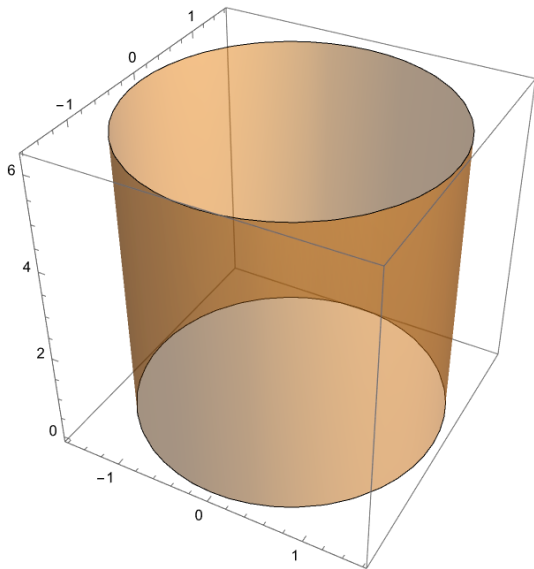
The AdS/CFT correspondence

The AdS/CFT correspondence is a conjectured relationship between two very different kinds of physical theories:

- Anti-de Sitter (AdS) space, which describes a type of spacetime with constant negative curvature, often used in theories of gravity like string theory.
- Conformal Field Theory (CFT), which is a quantum field theory with special symmetries, including scale invariance, that operates in flat spacetime.

The correspondence states that a CFT in d dimensions (the boundary theory) is equivalent to a theory of gravity in an AdS space of $d + 1$ dimensions (the bulk theory); that is the physics of the gravity theory in AdS space can be completely described by a quantum field theory without gravity on the boundary of that space.

Pictorial representation of $\text{AdS}_3/\text{CFT}_2$



The AdS/CFT correspondence resolves the black hole information paradox, at least to some extent, because it declares that the unitarity derived from the CFT wins the fight: black holes should evolve in a manner consistent with quantum mechanics. Black holes in the context of AdS/CFT correspond to a sliver of heavy states on the boundary CFT and since these states obey the usual rules of quantum mechanics (that is, they evolve unitarily) the black hole must also evolve in a unitary fashion, respecting the principles of quantum mechanics.

Understanding the actors

We will try to explore the black hole information paradox within the paradigm of the AdS/CFT correspondence. In particular, we will be looking at $\text{AdS}_3/\text{CFT}_2$ as:

1. BTZ black holes in AdS_3 can be used as a prototypical system to understand information losses.
2. Two dimensional CFTs afford us with a very well studied conformal group dictated by the Virasoro algebra.

With this in mind, let us now introduce the main terminology used for the later calculations.

Euclidean AdS_3 Geometry

Euclidean AdS_3 is defined as the hyperboloid

$$-u^2 + v^2 + x^2 + y^2 = -l^2 \quad (1)$$

embedded in $\mathbb{R}^{2,2}$, which is a pseudo-Riemannian manifold given by the metric

$$ds^2 = -du^2 + dv^2 + dx^2 + dy^2 \quad (2)$$

We can parametrize the embedding in the so-called *Euclidean global coordinates* $\{\tau, \rho, \theta\}$

$$\begin{aligned} u &= l \sin \tau \sec \rho & v &= l \cos \tau \sec \rho \\ x &= l \sin \theta \tan \rho & y &= l \cos \theta \tan \rho \end{aligned} \quad (3)$$

We get the Euclidean AdS_3 metric in terms of Euclidean global coordinates

$$ds^2 = \frac{l^2}{\cos^2 \rho} (d\tau^2 + d\rho^2 + \sin^2 \rho d\theta^2) \quad (4)$$

where the ranges of the coordinates are given by

$$0 \leq \rho < \frac{\pi}{2}, \quad 0 \leq \theta < 2\pi, \quad -\infty \leq \tau < +\infty \quad (5)$$

The BTZ black hole

A Poor Man's Schwarzschild

The BTZ (Baños-Teitelboim-Zanelli) black hole is a solution to Einstein's equations in a $2 + 1$ dimensional spacetime with a negative cosmological constant. With a metric that looks very similar to the Schwarzschild solution, it is used as a way to understand quantum theories of gravity in a lower dimension as it has many of the features of a Schwarzschild black hole: singularities, horizons; as well thermal properties in the Euclidean version. (Interestingly, it cannot give rise to gravitational waves)

Similar to how one uses the hydrogen atom as the prototypical system to understand the laws of quantum mechanics, the BTZ black hole is the standard system used to study various features of different theories of quantum gravity. Furthermore, since it is in a lower dimension it is ostensibly easier to do calculations. We will focus on this as a way to understand the black hole information paradox in the setting of $\text{AdS}_3/\text{CFT}_2$.

Understanding the Euclidean BTZ black hole

The static ($J = 0$) case

We use another set of coordinates, called the *Euclidean BTZ coordinates* as given below

$$\begin{aligned} &\text{Region I: } r_+ < r < \infty \\ &u = \sqrt{\tilde{r}^2} \cosh \tilde{\phi} \quad v = \sqrt{\tilde{r}^2 - l^2} \sin \tilde{t} \\ &x = \sqrt{\tilde{r}^2} \sinh \tilde{\phi} \quad y = \sqrt{\tilde{r}^2 - l^2} \cos \tilde{t} \end{aligned} \tag{6}$$

where

$$\tilde{r} = r \frac{l}{r_+} \quad \tilde{\phi} = \phi \frac{r_+}{l} \quad \tilde{t} = t \frac{r_+}{l^2} \tag{7}$$

The metric in these coordinates looks like

$$ds^2 = \left(\frac{r^2 - r_+^2}{l^2} \right) d\tau^2 + \left(\frac{r^2 - r_+^2}{l^2} \right)^{-1} dr^2 + r^2 d\phi^2 \quad (8)$$

with the coordinate ranges

$$r_+ \leq r < \infty, \quad -\infty < \phi < \infty \quad 0 \leq \tau < \frac{2\pi l^2}{r_+} \quad (9)$$

When we perform an identification of the t variable, we get the Euclidean BTZ black hole. Since it is locally AdS_3 , it solves Einstein's equations transitively. The thermality of the black hole is captured in the relationship

$$r_+ = 2\pi T_H \quad (10)$$

where T_H is the Hawking temperature.

Euclidean BTZ cigar shaped geometry

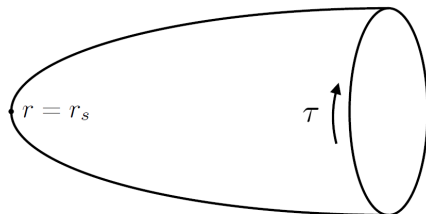


Figure: The BTZ black hole has a *topological* singularity rather than a *curvature* singularity as in Schwarzschild black holes; it is due to the identification of points on a smooth AdS_3 manifold that the conical singularity arises at $r = r_+$. In fact, we need to make time periodic so that there is no conical defect at the tip of the cigar, and it is interesting to note this in the context of thermality of black holes after Euclidean continuation.

Conformal Field Theory

Two dimensional conformal field theory (CFT₂)

A conformal group is the group of invertible transformations that leave the metric of d dimensional spacetime invariant up to a local scaling factor $\Lambda(x)$

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \Lambda(x)g_{\mu\nu}(x) \quad (11)$$

For two dimensions, it turns out that the change of coordinates $x \rightarrow x' = f(x)$ actually satisfy the Cauchy-Riemann conditions. This means that a conformal transformation in two dimensions is equivalent to changing the coordinates via a holomorphic function ($z \rightarrow f(z)$). This affords an infinite set of possible coordinate transformations and the analysis becomes that of complex functions.

Terminology

- Generators: these are the modes of the stress tensor $T_{zz}(z) = \sum_n z^{-2-n} L_n$ and satisfy the Virasoro algebra

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{m+n,0} \quad (12)$$

- Primary operators: These are fields that transform covariantly under conformal transformations $z \rightarrow w(z)$

$$\mathcal{O}(z, \bar{z}) \rightarrow \left(\frac{dw}{dz}\right)^h \left(\frac{d\bar{w}}{d\bar{z}}\right)^{\bar{h}} \mathcal{O}(w, \bar{w}) \quad (13)$$

where h is the holomorphic dimension. ($h \geq 0$)

- Scaling dimension $\Delta = h + \bar{h}$, Spin $s = h - \bar{h}$

- Central charge c : It is a parameter of the CFT under consideration, in our case, it is related to bulk theory as

$$c \equiv \frac{3}{2G_N} \quad (14)$$

and so it is effectively a measure of the coupling constant.
($c > 0$)

- State - Operator correspondence

$$|h\rangle = \mathcal{O}_h|0\rangle \quad (15)$$

A local operator/field creates a state with conformal dimension h .

- The Hilbert space tower is given by the combination of

$$L_0|h\rangle = h|h\rangle, \quad L_n|h\rangle = 0 \quad (n > 0) \quad L_{-n}|h\rangle = |h+n\rangle \quad (16)$$

AdS/CFT Correspondence Dictionary

AdS (Bulk Theory)	CFT (Boundary Theory)
3-dimensional AdS space	2-dimensional conformal field theory
Metric $g_{\mu\nu}$	Stress tensor $T_{\mu\nu}$
Bulk scalar field $\phi(\rho, \tau, \theta)$	Scalar operator $\mathcal{O}_h(z)$
Mass of bulk field $m_{\text{AdS}}^2 = \Delta(\Delta - 2)$	Scaling dimension Δ $\Delta = 2h$
Black hole in AdS space	Thermal state in CFT

Four Point Functions in CFT_2

The subject of interest is the vacuum four-point function of local scalar operators $\mathcal{O}_{h_i}(z_i)$ living in two dimensional Euclidean space (in other words, on the boundary).

$$\langle \mathcal{O}_{h_1}(z_1) \mathcal{O}_{h_2}(z_2) \mathcal{O}_{h_3}(z_3) \mathcal{O}_{h_4}(z_4) \rangle \xrightarrow{SO(3,1)} \langle \mathcal{O}_1(\infty) \mathcal{O}_2(1) \mathcal{O}_3(z, \bar{z}) \mathcal{O}_4(0) \rangle \quad (17)$$

where we have used the conformal symmetries to take three of the operators to specified locations.

A block decomposition expression can be made for this because the basis of the Hilbert space consists of primary operators $|\mathcal{O}_h\rangle$ and its descendants.

Virasoro block decomposition

The basis formed by these primary operators is the set of irreducible highest weight representations of the Virasoro algebra, and thus, we have a partial wave decomposition of the four point function

$$\langle \mathcal{O}_1(\infty) \mathcal{O}_2(1) \mathcal{O}_3(z, \bar{z}) \mathcal{O}_4(0) \rangle = \sum_{h, \bar{h}} P_{h, \bar{h}} \mathcal{V}_h(c, h_i, h; z) \mathcal{V}_{\bar{h}}(c, h_i, \bar{h}; \bar{z}) \quad (18)$$

This is called the Virasoro conformal block decomposition; if we had used the global conformal group then it would have been the global conformal block decomposition.

It would have been great if we had known the exact form of the Virasoro blocks; unfortunately a general closed form analytical solution is not known and thus, we have to work in certain approximation regimes.

It is already known that the $c \rightarrow \infty$ can be used to calculate semi-classical Virasoro blocks, as in this limit, they reduce to global conformal blocks when all the h_i are held fixed.

This can be extended to something called the “heavy-light semi-classical” regime; and it also has the desirable feature that it can be interpreted in a very natural way in AdS/CFT.

Relation to AdS/CFT

Via the AdS/CFT correspondence, the CFT stress tensor creates gravitons in AdS_3 ; and we also have the central charge c determining the strength of G_N interactions. The correlation functions can also be given an AdS interpretation; CFT states with $h_H \propto c$ correspond to BTZ black holes (or deficit angles). We aim to study the Virasoro blocks in the so-called heavy-light semi-classical limit wherein

$$\frac{h_H}{c}, h_L, h \text{ are fixed for } c \rightarrow \infty \quad (19)$$

The “light” operator $\mathcal{O}_L$ creates a state that can be thought of as a test mass probing the heavy AdS black hole (or deficit angle).

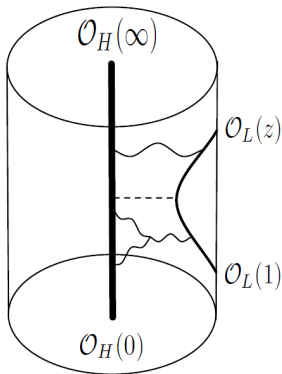


Figure: A pictorial representation of the four point correlator in the heavy-light limit; this represents the interaction between a light probe on the boundary and a black hole (heavy state) via gravitons.

What is known from the boundary theory?

This semiclassical approximation calculation has already been done from the boundary [2], and it is known that

$$\mathcal{V}_h(c, h_i, h; z) \mathcal{V}_{\bar{h}}(c, h_i, \bar{h}; \bar{z}) \propto \left[\sin \left(\frac{r_+ \tau}{2} \right) \right]^{2h-4h_L} {}_2F_1 \left(h, h; 2h; 1 - e^{ir_+ \tau} \right) {}_2F_1 \left(h, h; 2h; 1 - e^{-ir_+ \tau} \right) \quad (20)$$

We want to understand this from the bulk, that is, identify what the holographic dual of a conformal block; what is the geometric representation of this block in AdS.

Witten Diagrams

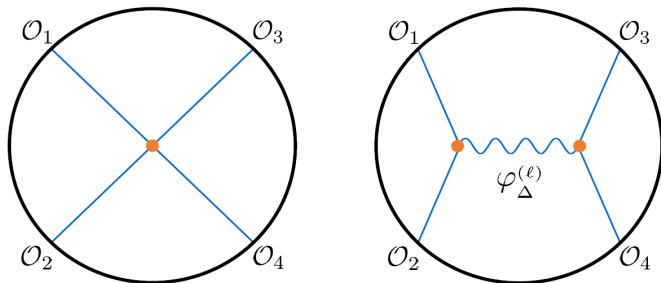


Figure: Tree level four point Witten digrams for scalar external operators; the one on the left has a contact interaction vertex while the one on the right has an intermediary.

The way these diagrams are calculated, is by noting the bulk-bulk and bulk-boundary propagators and then integrating them over the entirety of AdS.

For example, the one on the left is calculated by the formula

$$\int_y G_{b\partial}(y, x_1) G_{b\partial}(y, x_2) G_{b\partial}(y, x_3) G_{b\partial}(y, x_4) \quad (21)$$

while the one on the right is

$$\int_y \int_{y'} G_{b\partial}(y, x_1) G_{b\partial}(y, x_2) \times G_{bb}(y, y'; \Delta, l) \times G_{b\partial}(y', x_3) G_{b\partial}(y', x_4) \quad (22)$$

Geodesic Witten Diagrams

Holographic dual of the scalar conformal block

However, the conformal block itself only needs a geodesic Witten diagram [4]; and the relation between the diagram and the block is

$$W_{\Delta,l} \sim \mathcal{V}_h(c, h_i, h; z) \mathcal{V}_{\bar{h}}(c, h_i, \bar{h}; \bar{z}) \quad (23)$$

and, in particular, for a scalar conformal block (in the global limit $c \rightarrow \infty$, h_i fixed)

$$\begin{aligned} W_{2h,0} = & \int d\lambda \, G_{b\partial}(x_1, y(\lambda)) G_{b\partial}(x_2, y(\lambda)) \int d\lambda' \, G_{b\partial}(x_3, y(\lambda')) G_{b\partial}(x_4, y(\lambda')) \\ & \times G_{bb}(y(\lambda), y(\lambda'); 2h) \end{aligned} \quad (24)$$

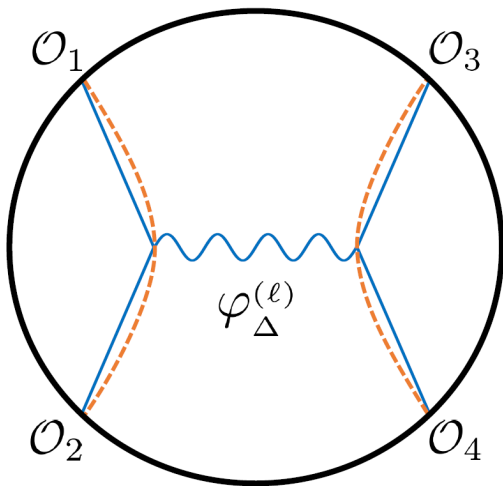


Figure: A geodesic Witten diagram

Calculating the conformal block in the bulk

Previous calculations [3] have shown that one can indeed reproduce Eq. 20 by setting up the following picture: the “heavy” geodesic backreacts and Eq. 24 is calculated in this new spacetime.

The backreaction leads to a metric given by (note $\alpha = 1$ gives global limit)

$$ds^2 = \frac{\alpha^2}{\cos^2 \rho} \left(\frac{d\rho^2}{\alpha^2} + d\tau^2 + \sin^2 \rho d\phi^2 \right) \quad (25)$$

For real $\alpha < 1$, this is a conical defect geometry with singularity at $\rho = 0$. For $\alpha^2 < 0$, it is a BTZ geometry. The idea is that the heavy geodesic is sourcing this new background and the light operator geodesic is probing the heavy operator geometry.

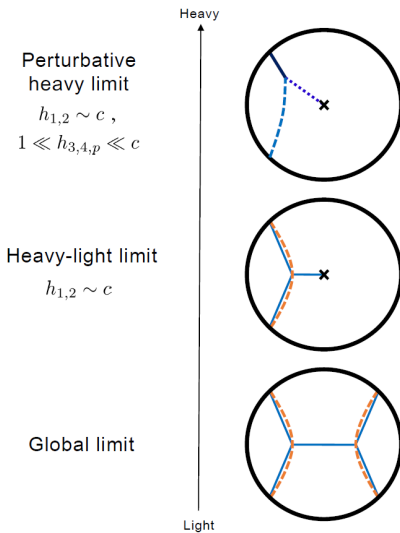


Figure: The various limits of the backreaction

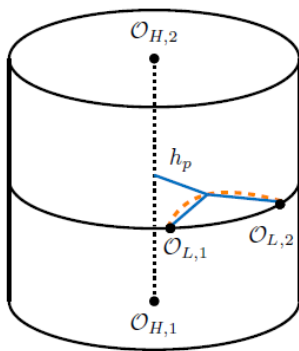


Figure: The set up for the Kraus calculation

- Heavy operator geodesic: Anchored at $\rho = 0$, extending from $\tau = -\infty$ to $\tau = +\infty$ on a fixed ϕ slice.
- Light operator geodesic: Connecting two boundary points $\rho \rightarrow \pi/2$ separated only along the angular coordinate ϕ , on a fixed time slice.
- To calculate, use global limit propagators and continue them using $\tau \rightarrow \alpha\tau, \phi \rightarrow \alpha\phi$
- The analytic continuation is used as a trick and it is not evident in the $\alpha^2 < 0$ case. The geodesics change character, there is no $\rho = 0$ singularity. There needs to be a more explicit calculation for the BTZ geometry.

- Metric:

$$ds^2 = \frac{\alpha^2}{\cos^2 \rho} \left(\frac{d\rho^2}{\alpha^2} + d\tau^2 + \sin^2 \rho d\phi^2 \right) \quad (26)$$

- Bulk-boundary propagator:

$$G_{b\partial}(x', y) = \left(\frac{\cos \rho}{\cosh(\tau - \tau') - \sin \rho \cos(\rho - \rho')} \right)^{2h} \quad (27)$$

- Bulk-bulk propagator:

$$G_{bb}(y, y'; 2h) = \xi^{2h} {}_2F_1\left(h, h + \frac{1}{2}; 2h; \xi^2\right) \quad (28)$$

with

$$\xi = \frac{\cos \rho \cos \rho'}{\cosh(\tau - \tau') - \sin \rho \sin \rho' \cos(\rho - \rho')} \quad (29)$$

Bulk prescription in explicit BTZ

Since we are dealing with the BTZ black hole geometry, we want the cleanest set up of the geodesic Witten diagram; and thus, we proceed to calculate all the relevant quantities in terms of BTZ coordinates. Note that since there is a complex continuation for the BTZ geometry from the Kraus calculation and also the fact that the backreaction does not start from $\rho = 0$ (there is in fact no $r < r_+$ region in BTZ), we need to set up the back reaction and the calculation from scratch.

Setting up the calculation

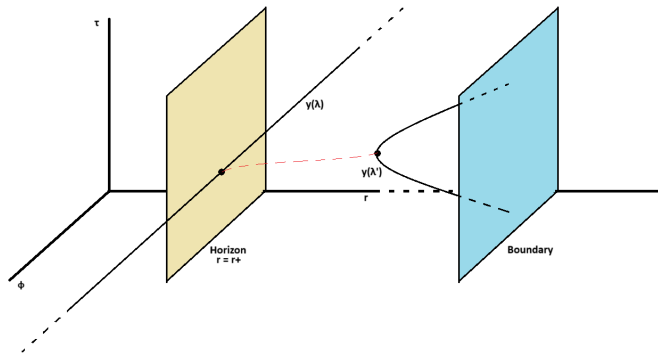


Figure: A bulk prescription to find the conformal block using a geodesic Witten diagram in BTZ geometry

The heavy operators backreact to set up the Euclidean black hole geometry, the source is at the to the horizon $r = r_+$.

$$\begin{aligned} x_1 &= (r = r_+, \phi = -\infty, \tau = \tau_0) \\ x_2 &= (r = r_+, \phi = +\infty, \tau = \tau_0) \end{aligned} \tag{30}$$

The light operators are on the boundary ($r \rightarrow \infty$), and they probe this geometry created by the back-reaction of the heavy operators.

$$\begin{aligned} x_3 &= (r' \rightarrow \infty, \tau = \tau_1, \phi = \phi_1) \\ x_4 &= (r' \rightarrow \infty, \tau = \tau_2, \phi = \phi_2) \end{aligned} \tag{31}$$

The probe interfacing between the heavy and light operators can be thought of as a field of conformal dimensions h being exchanged between the light and heavy operator geodesics.

Note the change from the Kraus calculation wherein they were placed at asymptotic times $\tau \rightarrow \pm\infty$, our prescription is slightly different; however, there is no conceptual problem with this owing to the democratic nature of τ and ϕ in terms of their coordinate ranges and behaviour in global Euclidean AdS_3 .

- Metric:

$$ds^2 = \frac{dr^2}{r^2 - r_+^2} + (r^2 - r_+^2)d\tau^2 + r^2 d\phi^2 \quad (32)$$

- Bulk-boundary propagator:

$$G_{b\partial}(x', y) = \left(\frac{r_+^2}{r \cosh(r_+ \Delta\phi) - \sqrt{r^2 - r_+^2} \cos(r_+ \Delta\tau)} \right)^{2h} \quad (33)$$

- Bulk-bulk propagator:

$$G_{bb}(y, y'; 2h) = \xi^{2h} {}_2F_1\left(h, h + \frac{1}{2}; 2h; \xi^2\right) \quad (34)$$

with

$$\xi = \frac{r_+^2}{rr' \cosh(r_+ \Delta\phi) - \sqrt{r^2 - r_+^2} \sqrt{r'^2 - r_+^2} \cos(r_+ \Delta\tau)}$$

Plugging these in Eq.24, we get

$$\mathcal{W}_{2h,0}(\tau) \sim x^{-4h_L+2h} \sum_{n=0}^{\infty} \frac{(h)_n (h + \frac{1}{2})_n}{(2h)_n} \frac{x^n}{n!} \int_{-\infty}^{+\infty} d\lambda (\cosh \lambda)^{-2n-2h} \int_{-\infty}^{+\infty} d\lambda' (\cosh \lambda')^{-2n-2h} \quad (36)$$

for

$$x = \sin \frac{r_+ \tau}{2} \quad (37)$$

which can be simplified to Eq.20 using identities. We have thus successfully realized a bulk construction to find the semi-classical conformal blocks in a BTZ black hole background!

Conformal blocks in $\text{AdS}_3/\text{CFT}_2$

The picture is that the heavy operators set up a classical asymptotically AdS_3 geometry corresponding to a BTZ black hole, and the light operators are described by a geodesic probing this background solution. It turns out that the thermal two-point correlator of the light probes/operators (h_L) in the background generated by the heavy operators (h_H)

$$\langle \mathcal{O}_L(1) \mathcal{O}_L(z, \bar{z}) \rangle_H = \langle \mathcal{O}_H(\infty) \mathcal{O}_L(1) \mathcal{O}_L(z, \bar{z}) \mathcal{O}_H(0) \rangle \quad (38)$$

Information Loss from Virasoro blocks

Semiclassical understanding of information loss (heavy-light limit)

Noting the universality of the vacuum Virasoro block, we check what Eq.20 gives for $h = 0$; we find that

$$\mathcal{W}_{2h,0}(\tau) \sim \left[\frac{1}{\sin(\pi T_H \tau)} \right]^{4h_L} \quad (39)$$

Thus, in Lorentzian time t_L , we find that the conformal block decays exponentially; in fact this is a sharp signature of information loss!

Thus, the semiclassical treatment gives us a pointed information loss; which we expected given Hawking's calculation. We will now try to see if there are any leads to where the information restoration happens.

Signatures of information restoration

When we plot the geodesics for our calculation, we see that they go closer and closer to the horizon. This is very indicative of the fact that near horizon effects will start to affect this particular geodesic Witten diagram calculation and it is unknown what after the geodesics “hit” the horizon when

$$\pi T_H \tau = \frac{\pi}{2} \implies \tau = \frac{\beta}{2} \quad (40)$$

This is interesting as previous research [1] on the CFT_2 end of the theory (reproduced in Fig. 8) shows that the deviations between the semiclassical and exact Virasoro blocks starts at $t_L = i\frac{\beta}{2}$, which is exactly what we expect based on our bulk prescription. This a very promising direction and we hope to find something fruitful when we explore this further.

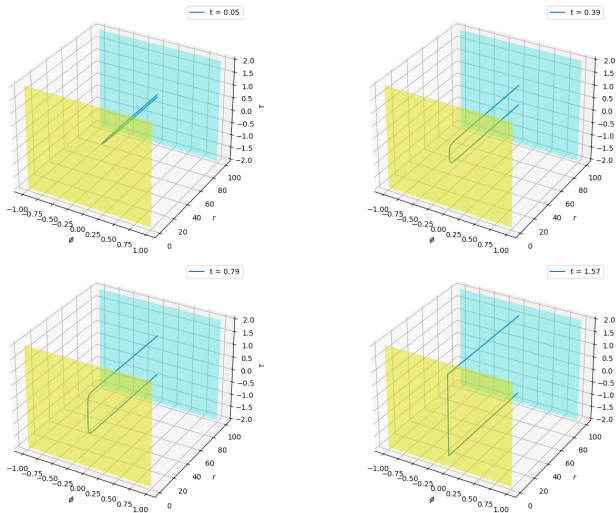


Figure: Geodesics in the geodesic Witten diagram set up

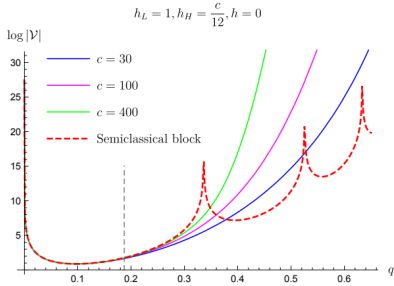


Figure 15. In this plot, we compare the exact and semiclassical blocks. One can see that at the positions of the semiclassical forbidden singularities, the exact blocks are smooth. Fixing h_L and $\frac{h_H}{c}$ as we increase c , the exact blocks approach the semiclassical block in the region between the origin and the first forbidden singularity. However, beyond the first forbidden singularity the exact blocks deviate greatly as we increase c . This indicates that we have passed a Stokes line (emanating from the forbidden singularity) and some other semiclassical saddle dominates the exact blocks in the large c limit. The gray line is the position of $t = \frac{i\beta}{2}$.

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Thank you!

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