

Limit Comparison Test:

If $\sum a_n$ and $\sum b_n$ are series with positive terms and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = C \quad \text{where } 0 < C < \infty,$$

then either both series converge or both diverge.

- If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ or ∞ , LCT is inconclusive $\Rightarrow$ pick a different b_n
OR use a different test.

- You can also do $\lim_{n \rightarrow \infty} \frac{b_n}{a_n} = \frac{1}{C}$

$$\swarrow \quad \frac{\pi/2}{\sqrt{n}} = b_n \quad \frac{1}{\sqrt{n}}$$

ex: $\sum_{n=1}^{\infty} \frac{\arctan(n)}{\sqrt{n}}$

← positive terms ✓

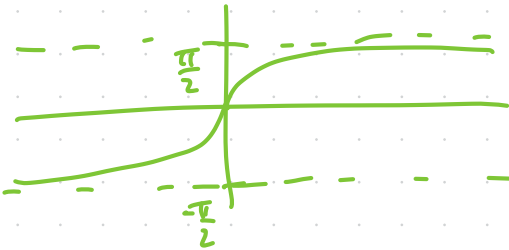
Pick $b_n = \frac{1}{\sqrt{n}} \geq 0$ ✓

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \quad p = \frac{1}{2} < 1$$

diverges

LCT: $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{\arctan(n)}{\cancel{\sqrt{n}}}}{\frac{1}{\cancel{\sqrt{n}}}}$

$$= \lim_{n \rightarrow \infty} \arctan(n) = \frac{\pi}{2}$$



Since $0 < \frac{\pi}{2} < \infty$, both series diverge.

$$\sum_{n=1}^{\infty} \frac{\arctan(n)}{\sqrt{n}} \text{ diverges by LCT.}$$

Guidelines

For choosing b_n :

① delete lower terms

② replacing complicated functions with one of their extreme values

For LCT, when choosing b_n , you need to match the speed at which the terms a_n go to 0. You can't under/overshoot in the same way as DCT.

ex:
$$\sum_{n=1}^{\infty} \frac{1}{e^n + n^2}$$

If using DCT, we could use

$$b_n = \frac{1}{e^n}$$

OR

$$b_n = \frac{1}{n^2}$$

$$\sum \left(\frac{1}{e}\right)^n \text{ converges}$$

$$\frac{1}{e^n + n^2} \leq \frac{1}{e^n} \quad \checkmark$$

$$\sum \frac{1}{n^2} \text{ converges}$$

$$\frac{1}{e^n + n^2} \leq \frac{1}{n^2}$$

If using LCT, only $b_n = \frac{1}{e^n}$ works

$$a_n = \frac{1}{e^n + n^2}$$

↑
dominant
term

Choice 1: $b_n = \frac{1}{n^2}$

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{e^n + n^2}}{\frac{1}{n^2}} \\&= \lim_{n \rightarrow \infty} \frac{n^2}{e^n + n^2} \\&= \lim_{n \rightarrow \infty} \frac{1}{\underbrace{\frac{e^n}{n^2}}_{\infty} + 1} = 0\end{aligned}$$

Choice 2: $b_n = \frac{1}{e^n}$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{e^n}{e^n + n^2} = 1$$

Growth Rates of Sequences (all go to ∞)

$$\ln(n) \ll \text{polynomial} \ll \text{exponential} \ll n! \ll n^n$$

$(n, n^2, n^2+n+1) \quad (e^n, 2^n)$ *

ex:

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 2 \ln(n)}$$

↑
dominant

ex:

$$\sum_{n=1}^{\infty} \frac{1}{n! + 2^n}$$

↑
dominant

When choosing b_n for LCT, constants matter less than DCT:

ex: $\sum_{n=1}^{\infty} \frac{3n-2}{2n^3+5}$

If using DCT, choose $b_n = \frac{3n}{2n^3} = \frac{3}{2} \cdot \frac{1}{n^2}$

If using LCT, can choose $b_n = \frac{1}{n^2}$.

ex: $\sum_{n=1}^{\infty} \frac{\arctan(n)}{\sqrt{n}}$

DCT: $b_n = \frac{\pi/2}{\sqrt{n}}$

LCT: $b_n = \frac{1}{\sqrt{n}}$