

ex: $\int_0^{\infty} \frac{1}{x^3} dx$

2 issues: ∞ in top bound
and discty in bottom
bound

$$= \underbrace{\int_0^1 \frac{1}{x^3} dx}_{(1)} + \underbrace{\int_1^{\infty} \frac{1}{x^3} dx}_{(2)}$$

$u = f(x)$
 $du = \frac{du}{dx} dx$
if $x=t, u=f(t)$

$$\int_0^1 \frac{1}{x^3} dx = \lim_{t \rightarrow 0^+} \int_t^1 x^{-3} dx$$



$$= \lim_{t \rightarrow 0^+} \left(-\frac{1}{2} x^{-2} \right) \Big|_t^1$$

$$= \lim_{t \rightarrow 0^+} \left(-\frac{1}{2} - \left(-\frac{1}{2} t^{-2} \right) \right)$$

$$= \lim_{t \rightarrow 0^+} \left(-\frac{1}{2} + \frac{1}{2t^2} \right)$$

$$= -\frac{1}{2} + \infty$$

$$= \infty \quad \text{diverges}$$

denom goes to 0,
so fraction
blows up

Since one piece diverges, the whole integral
diverges.

Fact: $\int_1^{\infty} \frac{1}{x^p} dx$ is convergent if $p > 1$
and divergent if $p \leq 1$.

examples using the fact

ex: $\int_1^{\infty} \frac{1}{x^2} dx$ $p = 2 > 1$ so converges

and $\int_1^{\infty} \frac{1}{x} dx$ $p = 1 \leq 1$ so diverges

ex: $\int_1^{\infty} \frac{1}{x\sqrt{x}} dx = \int_1^{\infty} \frac{1}{x^{3/2}} dx$

$p = \frac{3}{2} > 1 \Rightarrow \text{conv.}$

ex: $\int_1^{\infty} \frac{2}{\sqrt{x}} dx = 2 \int_1^{\infty} \frac{1}{x^{1/2}} dx$

$p = \frac{1}{2} \leq 1$
so diverges.

ex: $\int_5^{\infty} \frac{1}{\sqrt[4]{x}} dx$

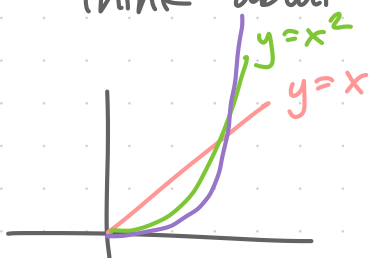
$$\underbrace{\int_1^{\infty} \frac{1}{x^{1/4}} dx}_{p=1/4 \leq 1 \text{ so diverges}} = \underbrace{\int_1^5 \frac{1}{x^{1/4}} dx}_{\text{a number}} + \underbrace{\int_5^{\infty} \frac{1}{x^{1/4}} dx}_{\text{diverges too}}$$

Now to prove:

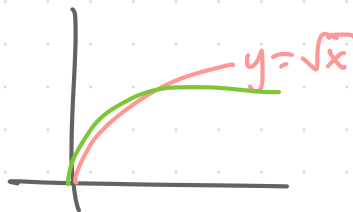
We'll want to compute $\lim_{x \rightarrow \infty} x^a$

where a is a constant.

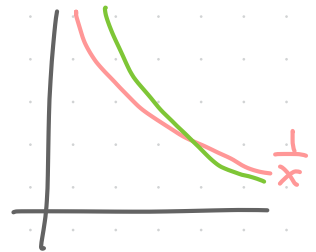
Think about the graph of x^a for $x \geq 0$



$a \geq 1$



$0 < a < 1$



$a < 0$

$$\lim_{x \rightarrow \infty} x^a = \infty$$

$$\text{if } a > 0$$

$$\lim_{x \rightarrow \infty} x^a = 0$$

$$\text{if } a < 0$$

Now we want to compute $\int_1^{\infty} \frac{1}{x^p} dx$.

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{t \rightarrow \infty} \int_1^t x^{-p} dx \quad \leftarrow \text{assume } p \neq 1$$

$$= \lim_{t \rightarrow \infty} \left(\frac{1}{1-p} x^{-p+1} \right) \Big|_1^t$$

$$= \lim_{t \rightarrow \infty} \left(\frac{1}{1-p} t^{1-p} - \frac{1}{1-p} \right)$$

$$= \lim_{t \rightarrow \infty} \frac{1}{1-p} (t^{1-p} - 1)$$

$$= \frac{1}{1-p} \lim_{t \rightarrow \infty} (t^{1-p} - 1)$$

Case 1: if $1-p > 0$, $\lim_{t \rightarrow \infty} t^{1-p} = \infty$

so $\int_1^{\infty} \frac{1}{x^p} dx$ diverges if $p < 1$

Case 2: if $1-p < 0$, $\lim_{t \rightarrow \infty} t^{1-p} = 0$

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^p} dx &= \frac{1}{1-p} \lim_{t \rightarrow \infty} (t^{1-p} - 1) \\ &= \frac{1}{1-p} (-1) \\ &= \frac{1}{p-1}\end{aligned}$$

↖ a constant
so the integral
converges

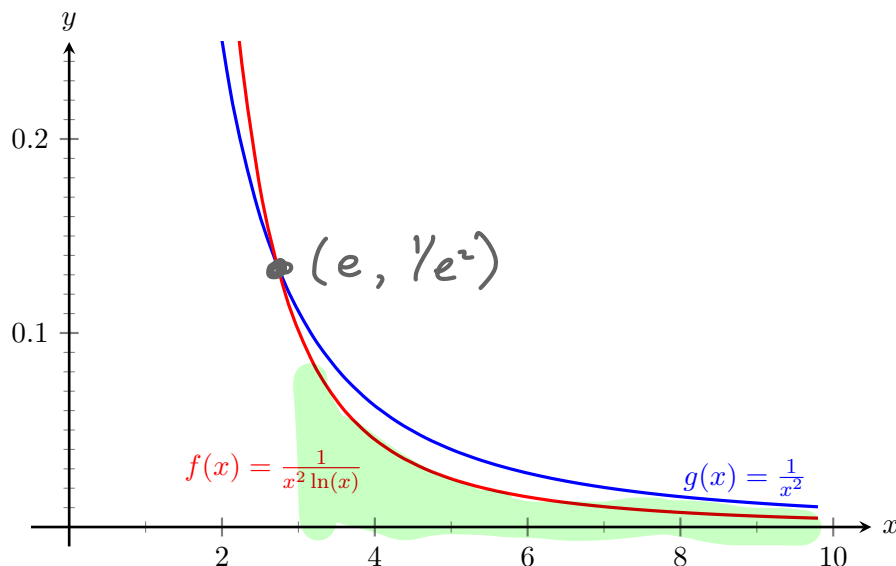
$\int_1^{\infty} \frac{1}{x^p} dx$ converges if $p > 1$

Case 3: if $p = 1$, $\int_1^{\infty} \frac{1}{x} dx$ diverges

(we showed before)

"Determine if the imp. integral converges or diverges. If it converges, compute the value." $\Rightarrow$ use the limit definition

1. Use technology to help you draw the graphs of $f(x) = \frac{1}{x^2 \ln(x)}$ and $g(x) = \frac{1}{x^2}$. Pay attention to any intersection points, and pay attention to which function has higher values.



2. Using the graph you just made:

(a) Do you think $\int_3^\infty \frac{1}{x^2 \ln(x)} dx \leq \int_3^\infty \frac{1}{x^2} dx$ is true? Why or why not?

Notice for $x \geq 3$,

$$\frac{1}{x^2 \ln x} \leq \frac{1}{x^2}$$

so this should be true.

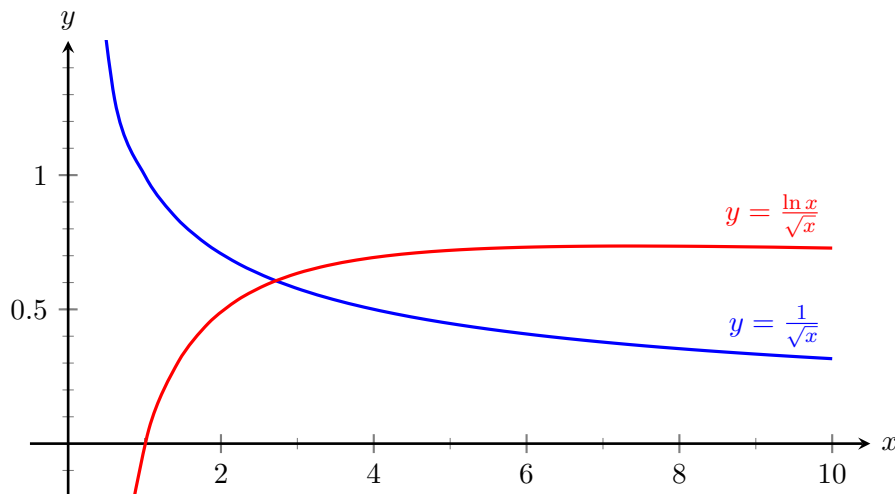
$1 \leq \ln x$
which is
true on
 $x \geq e$

- (b) What does this suggest about the convergence/divergence of the two integrals and why?

$\int_3^\infty \frac{1}{x^2} dx$ is a p-integral with $p=2 > 1$
so converges

$\Rightarrow \int_3^\infty \frac{1}{x^2 \ln x} dx$ also converges.

3. Use technology to help you draw the graphs of $f(x) = \frac{\ln(x)}{\sqrt{x}}$ and $g(x) = \frac{1}{\sqrt{x}}$. Do these functions both have horizontal asymptotes?

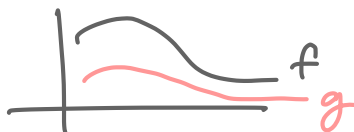


4. Using the graph you just made: *diverges* *also diverges*
- (a) Do you think $\int_3^{\infty} \frac{1}{\sqrt{x}} dx \leq \int_3^{\infty} \frac{\ln(x)}{\sqrt{x}} dx$ is true? Why or why not?

Notice $\frac{\ln x}{\sqrt{x}} \geq \frac{1}{\sqrt{x}}$ for all $x \geq e$

- (b) What does this suggest about the convergence/divergence of the two integrals and why?

$\int_3^{\infty} \frac{1}{x^{1/2}} dx$ diverges b/c $p = \frac{1}{2} < 1$

**The punchline:****Comparison Theorem for Integrals**

If f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$, then

(a) If $\int_a^\infty f(x) dx$ is convergent, then $\int_a^\infty g(x) dx$ is convergent.

(b) If $\int_a^\infty g(x) dx$ is divergent, then $\int_a^\infty f(x) dx$ is divergent.

If we know $\int_a^\infty f(x) dx$ diverges, what can we conclude about $\int_a^\infty g(x) dx$? *nothing*

If we know $\int_a^\infty g(x) dx$ converges, what can we conclude about $\int_a^\infty f(x) dx$? *nothing*

Putting it into practice:

5. Consider the integral $\int_2^\infty \frac{\cos^2(x)}{x^2} dx$.

(a) A good comparison function is: $\frac{1}{x^2}$

(b) What is the relationship between your comparison function and $\frac{\cos^2(x)}{x^2}$?

want: $\frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}$

$\cos^2 x \leq 1$ true b/c $\cos x \leq 1$

(c) What conclusion can you make using the Comparison Theorem?

$\int_2^\infty \frac{1}{x^2} dx$ converges b/c $p = 2 > 1$

by Comparison Thm. $\int_2^\infty \frac{\cos^2 x}{x^2} dx$ converges too

because $\frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}$ for all x

6. Consider the integral $\int_1^{\infty} \frac{1}{\sqrt{x^2 + 1}} dx$.

(a) A good comparison function is:

$$\frac{1}{\sqrt{x^2 + 1}} = \frac{1}{\sqrt{x^2}} = \frac{1}{x}$$

(b) What is the relationship between your comparison function and $\frac{1}{\sqrt{x^2 + 1}}$?

want: $\frac{1}{\sqrt{x^2}} \leq \frac{1}{\sqrt{x^2 + 1}}$

↑ larger denom than LHS
 $\Rightarrow$ smaller over all $\therefore$

(c) Will this inequality help you determine if $\int_1^{\infty} \frac{1}{\sqrt{x^2 + 1}} dx$ is convergent or divergent?

unfortunately, no

Guidelines for choosing a comparison function

① if you have a sum, only keep the fastest growing term

ex: $\int_1^{\infty} \frac{x+1}{\sqrt{x^4-1}} dx$

$$\frac{x+1}{\sqrt{x^4-1}} = \frac{x}{\sqrt{x^4}} = \frac{x}{x^2} = \frac{1}{x}$$

ex: $\int_2^{\infty} \frac{1}{x - \ln x} dx$

compare to

$$\frac{1}{x - \ln x} = \frac{1}{x}$$

fact: $\ln x \leq x$ *



$$\int_1^{\infty} \frac{1}{x} dx \text{ diverges b/c } p=1 \leq 1$$

want: $\frac{1}{x} \stackrel{?}{\leq} \frac{1}{x - \ln x}$

$$x(x - \ln x) \cdot \frac{1}{x} \leq \left(\frac{1}{x - \ln x} \right) \cdot x(x - \ln x)$$

$$x - \ln x \leq x$$

$$0 \leq \ln x \text{ true if } x \geq 1$$

ex: $\int_2^{\infty} \frac{1}{x+e^x} dx$

compare to $\frac{1}{x+e^x} = \frac{1}{e^x}$

would need to compute $\int_2^{\infty} \frac{1}{e^x} dx$

not a p-integral

∴

② replace functions with one their extreme values

ex: $\int_1^{\infty} \frac{1+\sin^2 x}{\sqrt{x}} dx$

since $0 \leq \sin^2 x \leq 1$,
replace $\sin^2 x$ with
either 0 or 1

$\frac{1}{\sqrt{x}}$

$\frac{2}{\sqrt{x}}$

$\int_1^{\infty} \frac{1}{\sqrt{x}} dx$ and $\int_1^{\infty} \frac{2}{\sqrt{x}} dx$ both diverge

because $p = 1/2 < 1$

want: $g(x) \leq \frac{1+\sin^2 x}{\sqrt{x}}$

so choose $g(x) = \frac{1}{\sqrt{x}}$

which makes this inequality true

ex: $\int_1^{\infty} \frac{1+\sin^2 x}{x^2} dx$

$g(x) = \frac{1}{x^2}$ or $\frac{2}{x^2}$

$\int_1^{\infty} g(x) dx$ converges
b/c $p=2 > 1$

want: $\frac{1+\sin^2 x}{x^2} \leq g(x)$

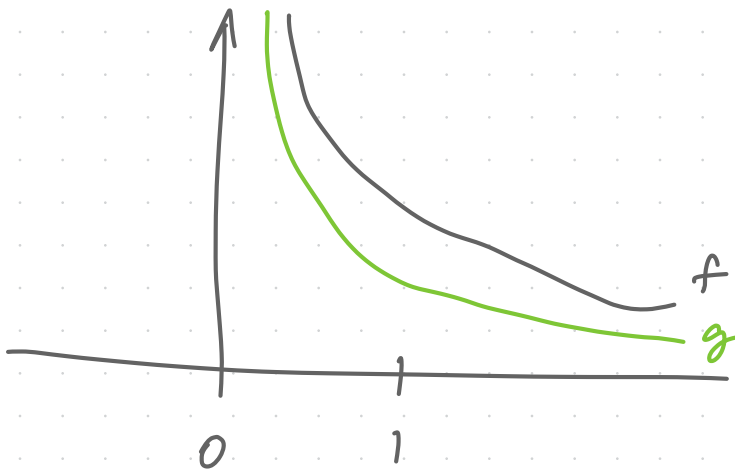
if $\int_1^{\infty} g(x) dx$ converges

then $\int_1^{\infty} \frac{1+\sin^2 x}{x^2} dx$ also convs.

③ replace $\ln x$ with 1

ex: $\int \frac{1}{x^2 \ln x} dx$

compare to $\frac{1}{x^2}$



$$\int_0^1 f(x) dx \text{ cogs} \Rightarrow \int_0^1 g(x) dx \text{ cogs}$$