

If  $f$  equal to its Taylor series? Not always...

To show that  $f$  is equal to its Taylor series at a point  $x$  (in the  $I(x)$ ), need to show

$$\lim_{n \rightarrow \infty} R_n(x) = 0$$

remember  
 $R_n(x) = f(x) - T_n(x)$   
If  $\lim_{n \rightarrow \infty} R_n(x) = 0$ ,  
$$f(x) = \underbrace{\lim_{n \rightarrow \infty} T_n(x)}_{\text{limit of partial sums of the Taylor series}}$$

$f(x) = \text{"Taylor series"}$

To do this, we use Taylor's Inequality

$$\left[ \begin{array}{l} \text{If } |f^{(n+1)}(x)| \leq M \text{ for } |x-a| \leq d, \text{ then} \\ |R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1} \end{array} \right]$$

ex: Show that  $f(x) = e^x$  is equal to its Maclaurin series at every  $x$  in  $\mathbb{R}$ .

$\hookrightarrow a=0$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{I.O.C is } (-\infty, \infty)$$

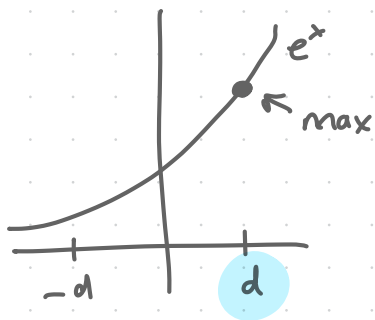
(Useful Fact:  $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$  for all  $x$ )

To use Taylor's Ineq., need to find  $M$

such that  $|f^{(n+1)}(x)| \leq M$  for  $|x-0| \leq d$   
 $|x| \leq d$

$[-d, d]$

$$f(x) = e^x \Rightarrow f^{(n+1)}(x) = e^x$$



Since  $f^{(n+1)}(x) = e^x$  is increasing on  $[-d, d]$ , the max occurs at the right endpoint:

$$|f^{(n+1)}(x)| \leq e^d \quad \text{for } |x| \leq d$$

Choose  $M = e^d$ .

Plug in to Taylor's Inequality:  $a=0$   
 $M=e^d$

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$$

$$|R_n(x)| \leq \frac{e^d}{(n+1)!} |x|^{n+1}$$

(Remember: want to show  $\lim_{n \rightarrow \infty} R_n(x) = 0$ )

$$\lim_{n \rightarrow \infty} |R_n(x)| \leq \lim_{n \rightarrow \infty} \left( \frac{e^d}{(n+1)!} |x|^{n+1} \right)$$

$$= e^d \lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!}$$

goes to 0, by fact above

$$= e^d \cdot 0 = 0$$

We've shown  $\lim_{n \rightarrow \infty} |R_n(x)| \leq 0$ .

This means  $\lim_{n \rightarrow \infty} |R_n(x)| = 0$ .

Also  $\lim_{n \rightarrow \infty} R_n(x) = 0$  because  $\lim_{n \rightarrow \infty} |R_n(x)| = 0$

(can show using the Squeeze Thm.)

Notice  $\lim_{n \rightarrow \infty} R_n(x) = 0$  no matter what  $x$  or  $a$  is!

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x.$$



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