

Important Ideas related to series:

$\sum_{n=1}^{\infty} a_n$  is defined as the limit of the partial sums  $\{S_n\}$

$$\sum_{n=1}^{\infty} a_n = \lim_{N \rightarrow \infty} S_N \quad \text{if this exists}$$

(if it DNE/ $\pm\infty$  we say the series diverges)

It's often difficult to compute the value of a series. Two notable exceptions:

- telescoping series (using  $S_n$ )
- geometric series (using  $\frac{\text{first term}}{1-r}$ )

For most other series, can only tell if they converge or diverge (can't compute the value).

For this we have our series tests:

- Divergence Test
- Geometric Series Test
- Integral Test
- DCT/LCT
- Alt. Series Test (AST)

## telescoping example

$$\text{ex: } \sum_{n=3}^{\infty} \frac{1}{n^2 + 5n + 6} = \sum_{n=3}^{\infty} \underbrace{\frac{1}{n+2} - \frac{1}{n+3}}_{\text{PFD}}$$

$$\left\{ \begin{aligned} S_1 &= \frac{1}{5} - \frac{1}{6} \\ S_2 &= \frac{1}{5} - \cancel{\frac{1}{6}} + \cancel{\frac{1}{6}} - \frac{1}{7} = \frac{1}{5} - \frac{1}{7} \\ S_3 &= \frac{1}{5} - \cancel{\frac{1}{6}} + \cancel{\frac{1}{6}} - \cancel{\frac{1}{7}} + \cancel{\frac{1}{7}} - \frac{1}{8} = \frac{1}{5} - \frac{1}{8} \\ &\vdots \end{aligned} \right.$$

$$S_n = \frac{1}{5} - \frac{1}{n+5}$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1}{5} - \frac{1}{n+5} = \frac{1}{5}$$

The series converges and its value is  $\frac{1}{5}$ .

example: (similar to HW8 #4)

The meaning of the decimal representation of a number  $0.d_1d_2d_3d_4\ldots$  where each  $d_n$  is a number  $0, 1, 2, \ldots, 9$  is

$$0.d_1d_2d_3d_4\ldots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{d_4}{10^4} + \ldots = \sum_{n=1}^{\infty} \frac{d_n}{10^n}$$

Show that this series always converges.

Comparison Test: For each  $n$ ,  $d_n \leq 9$

Also terms  
are all positive

so  $\frac{d_n}{10^n} \leq \frac{9}{10^n}$  ✓

Choose  $b_n = \frac{9}{10^n}$ ,  $\geq 0$

Then  $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 9 \left(\frac{1}{10}\right)^n$   
geometric  
 $r = \frac{1}{10}$

so converges  
because  $|r| < 1$

So  $\sum_{n=1}^{\infty} \frac{d_n}{10^n}$  also converges by  
Comparison Test.

## Alternating Series Test

Suppose  $\sum a_n$  is alternating and  $b_n = |a_n|$ .

If:

①  $\lim_{n \rightarrow \infty} b_n = 0$

②  $\{b_n\}$  is decreasing

then the series  $\sum a_n$  converges.

Note: If ① or ② is not true, the test fails.

If that happens, try the Divergence Test.

ex:  $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\ln n}$

$$(-1)^{n+1} = (-1)^{n-1} \underbrace{(-1)^2}_1$$

$$b_n = \frac{1}{\ln n}$$

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0 \quad \checkmark$$

$\{b_n\}$  is decreasing <sup>✓</sup> because  $\ln n$  is increasing so decreasing b/c 1 over increasing

$$\left( \text{or } \frac{1}{\ln(n+1)} < \frac{1}{\ln n} \text{ for all } n \right)$$

So the series converges by AST.

ex:  $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1} \leftarrow a_n = (-1)^n \frac{n}{n+1}$

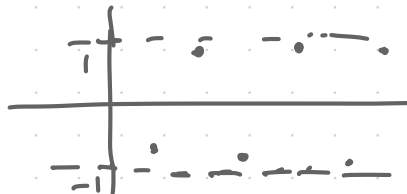
AST:  $b_n = \frac{n}{n+1}$

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$$

AST fails  
 $\Rightarrow$  try Divergence Test

Divergence Test:

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \underbrace{(-1)^n}_{\text{alternating}} \underbrace{\frac{n}{n+1}}_{\text{goes to 1}} = \text{DNE}$$



limit DNE

Since  $\lim_{n \rightarrow \infty} a_n \neq 0$  the series diverges by the Divergence Test.

ex:  $\sum_{n=1}^{\infty} \frac{n \cos(n\pi)}{2^n}$

Alternating?

$n=1$

$\cos(n\pi) = \cos(\pi) = -1$

$n=2$

$\cos(2\pi) = 1$

$n=3$

$\cos(3\pi) = -1$

$n=4$

$\cos(4\pi) = 1$

$(-1)^n$  or  $(-1)^{n+1}$

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{2^n}$$

AST:  $b_n = \frac{n}{2^n}$

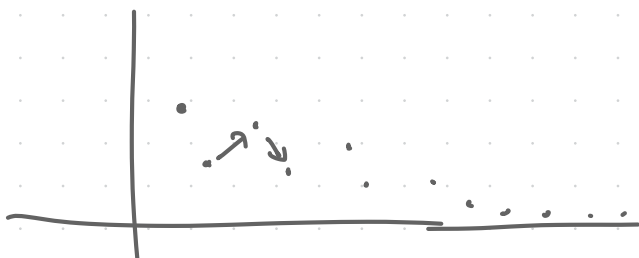
$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{n}{2^n} \stackrel{LH}{=} \lim_{n \rightarrow \infty} \frac{1}{2^n \cdot \ln 2} = 0$

$b_n$  decreasing?

$$b_{n+1} < b_n$$

$$\frac{n+1}{2^{n+1}} < \frac{n}{2^n} \quad ?$$

hard to see  
that it's true,  
so take the  
derivative



$$b_n = \frac{n}{2^n} \rightarrow f(x) = \frac{x}{2^x}$$

showing  
decreasing  
by using  
the derivative

$$f'(x) = \frac{2^x - x \cdot \ln 2 \cdot 2^x}{(2^x)^2}$$

$$= \frac{2^x (1 - \ln 2 \cdot x)}{(2^x)^2} < 0$$

$f'(x) < 0$  when this is true  $\rightarrow$

$$\boxed{1 - \ln 2 \cdot x < 0}$$

$$1 < \ln 2 \cdot x$$

$$\frac{1}{\ln 2} < x$$

decreasing on  $(\frac{1}{\ln 2}, \infty)$  ✓

So series converges by AST.

---

|       |                                       |                                |                                                      |
|-------|---------------------------------------|--------------------------------|------------------------------------------------------|
| Plan: | <u>Wed</u><br>absolute<br>convergence | <u>Thurs.</u><br>Ratio<br>Test | <u>Fri</u><br>series practice<br>remainder estimates |
|-------|---------------------------------------|--------------------------------|------------------------------------------------------|

---

## 8.4 - Absolute vs. Conditional Convergence

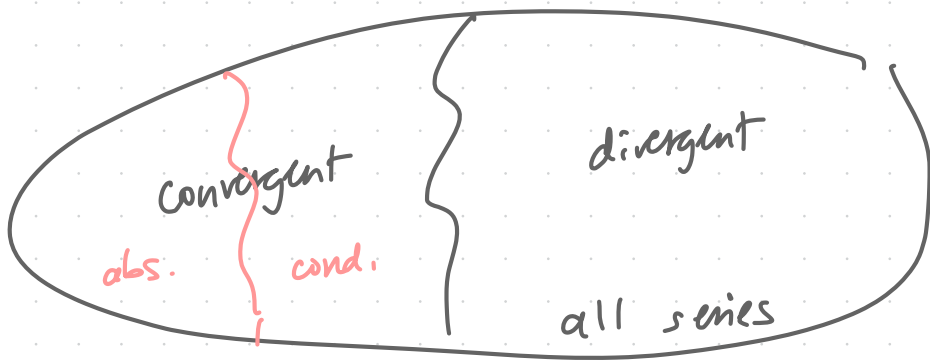
Defn:

A series is absolutely convergent if

$\sum |a_n|$  is convergent.

A series is conditionally convergent if it is convergent but not absolutely convergent

( $\sum a_n$  converges but  $\sum |a_n|$  diverges)



If your series has all positive terms, it's automatically absolutely convergent (assuming it's convergent).