Lipschitz Geometry of Mixed Pham-Brieskorn Singularities

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Abstract

We establish conditions for topological and bi-Lipschitz equivalences within a class of mixed singularities of Pham-Brieskorn type. We then derive the construction of topologically trivial infinite families with distinct bi-Lipschitz types. We also investigate the same problem applied to the associated mixed surfaces and deduce conditions for inner, outer, and ambient bi-Lipschitz equivalences. This is a first step to understanding the bi-Lipschitz properties of mixed functions and varieties through the relations with well-established complex cases and their simpler representatives.

Introduction

- Mixed functions are real analytic maps $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ on complex variables $z = (z_1, \ldots, z_n)$ and their conjugates $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n)$.
- We study the family of mixed function germs of the form

$$f_{\mathbf{a},\mathbf{b}} = z_1^{a_1+b_1}\bar{z}_1^{b_1} + \cdots + z_n^{a_n+b_n}\bar{z}_n^{b_n},$$

  where $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots, b_n)$ are vectors of non-negative integers such that $a_i + 2b_i \leq a_j + 2b_j$ for all $i < j$. We call $f_{\mathbf{a},\mathbf{b}}$ a **mixed Pham-Brieskorn polynomial**.
- The mixed polynomials $f_{\mathbf{a},\mathbf{b}}$ were first studied by Seade [8] and Seade-Ruas-Verjovsky [6] as examples of real isolated singularities admitting a spherical Milnor fibration with projection map in the usual form $f_{\mathbf{a},\mathbf{b}}/\|f_{\mathbf{a},\mathbf{b}}\|$.
- As a natural generalization of holomorphic functions, mixed functions inherit several properties from this context and play a prominent role in the area, being objects of intense research.
- Our primary objective here is to investigate mixed singularities and varieties from the Lipschitz geometry point of view. As exemplified by the characteristics of the family above, the invariant properties by Lipschitz equivalence lie between the topological and analytic realms, justifying this approach.
- Throughout this work, $\mathbb{F} = \mathbb{R}, \mathbb{C}$.

Definition 1. *Two map germs $f, g : (\mathbb{F}^n, 0) \longrightarrow (\mathbb{F}^p, 0)$ are* **bi-Lipschitz equivalent** *if there exists a Lipschitz map germ $\phi : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ whose inverse is Lipschitz and $f \circ \phi = g$. We refer to* **topological equivalence** *if ϕ is only a homeomorphism.*

Topological Equivalence

The first relation between mixed and complex singularities is given by the following proposition.

Proposition 2 (Theorem 4.1, [6]). *Let $b_i \geq 0$ for all i. The following statements hold:*

1. *If $a_i \geq 1$ for all i, then $f_{\mathbf{a},\mathbf{b}}$ is topologically equivalent to the holomorphic Pham-Brieskorn polynomial $g_{\mathbf{a}}$.*
2. *If $a_i = 0$ for all $i = 1, \ldots, k$ and $a_i \geq 1$ for all $i = k+1, \ldots, n$, then $f_{\mathbf{a},\mathbf{b}}$ is topologically equivalent to $g_{\mathbf{a}} = \sum_{i=1}^{k} \|z_i\|^{2b_i} + \sum_{i=k+1}^{n} z_i^{a_i}$, where $1 \leq k < n$.*

We call $g_{\mathbf{a}}$ the **topological normal form** *of $f_{\mathbf{a},\mathbf{b}}$.*

On the other hand, the topological classification of Pham-Brieskorn singularities is determined by the next result.

Theorem 3 (Theorem, [9]). *Let $g_{\mathbf{a}}, g_{\mathbf{c}} : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ be two polynomial germs of Pham-Brieskorn type, where $a_i, c_i \geq 2$ for all i. Then the hypersurface germs defined by $g_{\mathbf{a}}$ and $g_{\mathbf{c}}$ have the same topological type if and only if $\mathbf{a} = \mathbf{c}$.*

We can completely describe now the topological type of the family.

Theorem 4. *Let $b_i, d_i \geq 0$ for all i and suppose that $f_{\mathbf{a},\mathbf{b}}$ is topologically equivalent to $f_{\mathbf{c},\mathbf{d}}$. The following statements hold:*

1. *If $a_i, c_i \geq 2$, then $\mathbf{a} = \mathbf{c}$.*
2. *If $a_i = 0$ for all $i = 1, \ldots, k$ and $a_i \geq 2$ for all $i = k+1, \ldots, n$, then $\mathbf{a} = \mathbf{c}$.*
3. *If $a_i = 1$ for some i, then $c_j = 1$ for some j.*

Multiplicity

The first invariant of the bi-Lipshitz equivalence of singularities of function germs is the multiplicity.

Lemma 5 (Theorem 5, [4]). *Let $F, G : (\mathbb{F}^n \times \mathbb{F}^p, 0) \longrightarrow (\mathbb{F}, 0)$ be two smooth germs of the following forms: $F(x, y) = P_F(x) + f(y)$ and $G(x, y) = P_G(x) + g(y)$ such that:*

1. *$P_F(x), P_G(x) : (\mathbb{F}^n, 0) \longrightarrow (\mathbb{F}, 0)$ are homogeneous polynomials and have isolated singularity at the origin.*
2. *$f(y), g(y) : (\mathbb{F}^p, 0) \longrightarrow (\mathbb{F}, 0)$ are smooth germs with $m_f > m_{P_F} \geq 2$ and $m_g > m_{P_G} \geq 2$.*

If F and G are bi-Lipschitz equivalent, then $m_f = m_g$. Moreover, the singular sets Σ_{H_f} and Σ_{H_g} of H_f and H_g, respectively, are ambient bi-Lipschitz equivalent.

The next theorem gives a generalization of the previous result.

Theorem 6. *Let $F, G : (\mathbb{F}^{2n_1} \times \cdots \times \mathbb{F}^{2n_k}, 0) \longrightarrow (\mathbb{F}, 0)$ be a pair of smooth function germs of the following forms:*

$$\begin{cases} F(x_1, y_1, \ldots, x_k, y_k) = f_1(x_1, y_1) + \cdots + f_k(x_k, y_k) \\ G(x_1, y_1, \ldots, x_k, y_k) = g_1(x_1, y_1) + \cdots + g_k(x_k, y_k) \end{cases}$$

Suppose that:

1. *For each i, $f_i(x_i, y_i), g_i(x_i, y_i) : (\mathbb{F}^{2n_i}, 0) \longrightarrow (\mathbb{F}, 0)$ are homogeneous polynomials and have isolated singularity at the origin.*
2. *$2 \leq m_{f_1} < m_{f_2} < \cdots < m_{f_k}$ and $2 \leq m_{g_1} < m_{g_2} < \cdots < m_{g_k}$.*

If F and G are bi-Lipschitz equivalent, then $m_{f_i} = m_{g_i}$ for all $i = 1, \ldots, k$.

Bi-Lipschitz Equivalence

The topological type of $f_{\mathbf{a},\mathbf{b}}$ is determined by the exponents $\mathbf{a}$. On the other hand, the multiplicity is an invariant of the bi-Lipschitz equivalence. This leads to the following conclusion.

Theorem 7. *Let $b_i, d_i \geq 0$ for all i and suppose that $f_{\mathbf{a},\mathbf{b}}$ is bi-Lipschitz equivalent to $f_{\mathbf{c},\mathbf{d}}$. The following statements hold:*

1. *If $a_i, c_i \geq 2$, then $(\mathbf{a}, \mathbf{b}) = (\mathbf{c}, \mathbf{d})$.*
2. *If $a_i = 0$ for all $i = 1, \ldots, k$ and $a_i \geq 2$ for all $i = k+1, \ldots, n$, then $(\mathbf{a}, \mathbf{b}) = (\mathbf{c}, \mathbf{d})$.*
3. *If $a_i = 1$ for some i, then $c_j = 1$ for some j. Moreover, $a_i + 2b_i = c_i + 2d_i$ for all i.*

This implies the existence of families with the same topological, but distinct Lipschitz types.

Corollary 8. *Let $\mathbf{a}_0$ be a fixed vector of non-negative integers, where $a_i \geq 0$ for all i and $a_j \geq 1$ for some j. The family $\Gamma_{\mathbf{a}_0} = \{f_{\mathbf{a}_0,\mathbf{b}} : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0) : b_1, \ldots, b_n \in \mathbb{N}\}$ is topologically trivial but contains infinitely many distinct Lipschitz classes.*

Mixed Surfaces

The above question applied for mixed varieties reads as:

Let $X_{\mathbf{a},\mathbf{b}}$ and $X_{\mathbf{c},\mathbf{d}}$ be the **set germs** *defined by $f_{\mathbf{a},\mathbf{b}}$ and $f_{\mathbf{c},\mathbf{d}}$, respectively. What are the conditions on $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ for $X_{\mathbf{a},\mathbf{b}}$ and $X_{\mathbf{c},\mathbf{d}}$ to be bi-Lipschitz equivalent in relation to the inner, outer, or ambient metrics?*

We address this problem in the case $n = 2$. For this purpose, we determine the mixed tangent cones and use the geometric invariance of these objects under bi-Lipschitz homeomorphisms.

Theorem 9 (Theorem 3.2, [7]). *Let $X, Y \subset \mathbb{R}^n$ be subanalytic sets. If the germs (X, x_0) and (Y, y_0) are outer bi-Lipschitz equivalent, then the tangent cones $(C(X, x_0), x_0)$ and $(C(Y, y_0), y_0)$ are also outer bi-Lipschitz equivalent.*

Proposition 10. *Let $a_i \geq 1$, $b_i \geq 0$ for $i = 1, 2$. Then the mixed surface germ $X_{\mathbf{a},\mathbf{b}}$ is* **inner** *bi-Lipschitz equivalent to the metric cone H_1.*

Proposition 11. *Let $b_i, d_i \geq 0$ for $i = 1, 2$ and suppose that $X_{\mathbf{a},\mathbf{b}}$ and $X_{\mathbf{c},\mathbf{d}}$ are* **outer** *bi-Lipschitz equivalent. The following statements hold:*

1. *If $a_1 \geq 1$ and $a_1 + 2b_1 < a_2 + 2b_2$, then $c_1 \geq 1$ and $c_1 + 2d_1 < c_2 + 2d_2$.*
2. *If $a_1 \geq 1$ and $a_1 + 2b_1 = a_2 + 2b_2$, then $c_1 + 2d_1 = c_2 + 2d_2$.*
3. *If $a_1 = 0$ and $2b_1 < a_2 + 2b_2$, then $c_1 = 0$ and $2d_1 < c_1 + 2d_2$.*
4. *If $a_1 = 0$ and $2b_1 = a_2 + 2b_2$, then $c_1 + 2d_1 = c_2 + 2d_2$.*

Proposition 12. *Let $a_i, c_i \geq 2$, $b_i, d_i \geq 0$ for $i = 1, 2$ and suppose that $X_{\mathbf{a},\mathbf{b}}$ and $X_{\mathbf{c},\mathbf{d}}$ are* **ambient** *bi-Lipschitz equivalent. It holds that $\mathbf{a} = \mathbf{c}$ and*

1. *If $a_1 + 2b_1 < a_2 + 2b_2$, then $c_1 + 2d_1 < c_2 + 2d_2$.*
2. *If $a_1 + 2b_1 = a_2 + 2b_2$, then $c_1 + 2d_1 = c_2 + 2d_2$.*

The ambient geometry of $X_{\mathbf{a},\mathbf{b}}$ is determined by $X_{\mathbf{a}}$ as follows.

Proposition 13. *Let $a_i \geq 1$ and $b_i \geq 0$ for $i = 1, 2$. If $X_{\mathbf{a},\mathbf{b}} \subset \mathbb{C}^2$ is ambient bi-Lipschitz equivalent to a complex analytic curve $h^{-1}(0) \subset \mathbb{C}^2$, where $h : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ is a complex analytic function germ, then $X_{\mathbf{a},\mathbf{b}}$ and $X_{\mathbf{a}}$ are outer bi-Lipschitz equivalent.*

Remark 14. Recently, J. Edson Sampaio, A. Fernandes, and Z. Jelonek have proved the following result in [3]. If the pair $X_{\mathbf{a}}, X_{\mathbf{c}} \subset \mathbb{C}^n$ of Pham-Brieskorn hypersurfaces is bi-Lipschitz equivalent, then their multiplicities coincide.

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