

Absolute vs. Conditional Convergence

Defns:

A series $\sum a_n$ is absolutely convergent if $\sum |a_n|$ is convergent.

A series $\sum a_n$ is conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges (i.e. the series is convergent but not absolutely convergent)

ex: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

← converges, AST: $b_n = \frac{1}{n} \rightarrow 0$

and decreasing
 $\left(\frac{1}{n+1} < \frac{1}{n}\right)$

two different series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

← diverges, harmonic series

The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

ex: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

← converges by **AST**
(fill in details yourself)

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad p=2 > 1$$

so converges

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent.

Fact: If a series is absolutely convergent,
then it's convergent.

to prove: Companion Test with
 $\sum a_n$ and $\sum |a_n|$

ex: $\sum_{n=1}^{\infty} \frac{\sin(4n)}{4^n}$ converges or diverges?

We'd want to use Comparison Test
but terms are not positive...

Plan: show its absolutely convergent
so "regular" convergent.

$$\sum_{n=1}^{\infty} \left| \frac{\sin(4n)}{4^n} \right|$$

← positive terms ✓

Comp Test: Let $a_n = \left| \frac{\sin(4n)}{4^n} \right|$,

Choose $b_n = \frac{1}{4^n} \geq 0$ ✓

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$$

← geometric,
 $|r| = \frac{1}{4} < 1$

converges.

Want: $a_n \leq b_n$

$$\left| \frac{\sin(4n)}{4^n} \right| \leq \frac{1}{4^n}$$

← true because
 $|\sin(4n)| \leq 1$ ✓

$$\sum_{n=1}^{\infty} \left| \frac{\sin(4n)}{4^n} \right|$$

converges by

Comparison Test.

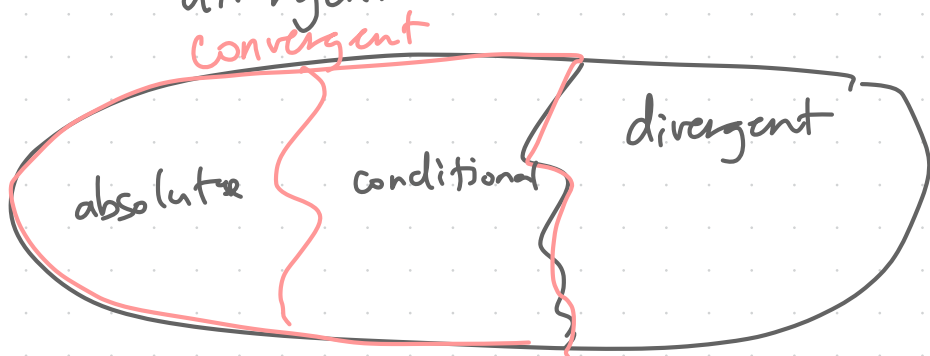
which means that

$$\sum_{n=1}^{\infty} \frac{\sin(4n)}{4^n} \text{ is}$$

absolutely convergent
hence convergent in
the normal sense.

Any series falls into exactly one of the following categories:

- absolutely convergent
- conditionally convergent
- divergent



example: Determine if the series converges absolutely, converges conditionally or ~~diverges~~.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{6n^2+3}$$

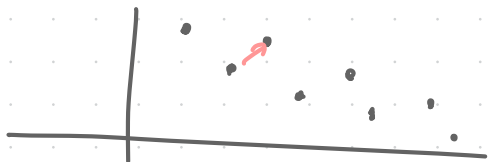
AST: $b_n = \frac{1}{6n^2+3}$

← not like choosing b_n in Comp. Test. Needs to be the part without the alternating.

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{6n^2+3} = 0 \quad \checkmark$$

b_n decreasing?

$$b_{n+1} \leq b_n \text{ for ALL } n$$



$$\frac{1}{6(n+1)^2+3} \leq \frac{1}{6n^2+3} \quad \checkmark$$

↑ larger denominator

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{6n^2+3} \quad \text{converges by AST.}$$

Now absolute vs. conditional:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{6n^2+3} \right| = \sum_{n=1}^{\infty} \frac{1}{6n^2+3}$$

(Limit) Comparison: $a_n = \frac{1}{6n^2+3} \geq 0$

Choose $b_n = \frac{1}{n^2} \geq 0$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$p=2 > 1 \leftarrow$
so converges ✓

LCT: $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1/(6n^2+3)}{1/n^2}$

$$= \lim_{n \rightarrow \infty} \frac{n^2}{6n^2+3} = \frac{1}{6} < \infty$$

Want this to be a finite positive value $\nearrow$

$\nwarrow$ include!

Since $0 < \frac{1}{6} < \infty$, both series converge by Limit Comparison Test.

Conclusion: The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{6n^2+3}$ converges absolutely.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{6n^2+3}$ convs and $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{6n^2+3} \right|$ convs

unnecessary

example: $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$ (same instructions)

Scratch work: $\sum_{n=2}^{\infty} \left| (-1)^n \frac{\ln n}{n} \right| = \sum_{n=2}^{\infty} \frac{\ln n}{n}$

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} \stackrel{LH}{=} \lim_{n \rightarrow \infty} \frac{1/n}{1} = 0$$

cvgt = convergent
convs = converges

Div. Test inconclusive

We suspect it is not absolutely convergent.

(cond. cvgt), (div.)?

$$\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$$

AST: $b_n = \frac{\ln n}{n}$

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{\ln n}{n} < \frac{\infty}{\infty}$$

$$\stackrel{\text{LH}}{=} \lim_{n \rightarrow \infty} \frac{1/n}{1} = 0 \quad \checkmark$$

b_n decreasing?

$$f(x) = \frac{\ln x}{x}$$

$$f'(x) = \frac{\frac{1}{x} \cdot x - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2} < 0$$

$$1 - \ln x < 0$$

$$1 < \ln x$$

$$2.7 \approx e < x$$

$\Rightarrow b_n$ is decreasing for $n \geq 3$

good enough $\checkmark$

$\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$ converges by AST.

$$\sum_{n=2}^{\infty} \left| (-1)^n \frac{\ln n}{n} \right| = \sum_{n=2}^{\infty} \frac{\ln n}{n}$$

Comparison Test: $a_n = \frac{\ln n}{n} \geq 0$

Choose $b_n = \frac{1}{n} \geq 0$.

$$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n}$$

diverges,
harmonic

want: $b_n \leq a_n$

$$\frac{1}{n} \leq \frac{\ln n}{n}$$

$$1 \leq \ln n$$

$$e \leq n$$

the inequality is true for $n \geq 3$, which is ✓
good enough.

$\sum_{n=2}^{\infty} |a_n| = \sum_{n=2}^{\infty} \frac{\ln n}{n}$ diverges by the Comparison Test.

Conclusion: $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$ is conditionally convergent.