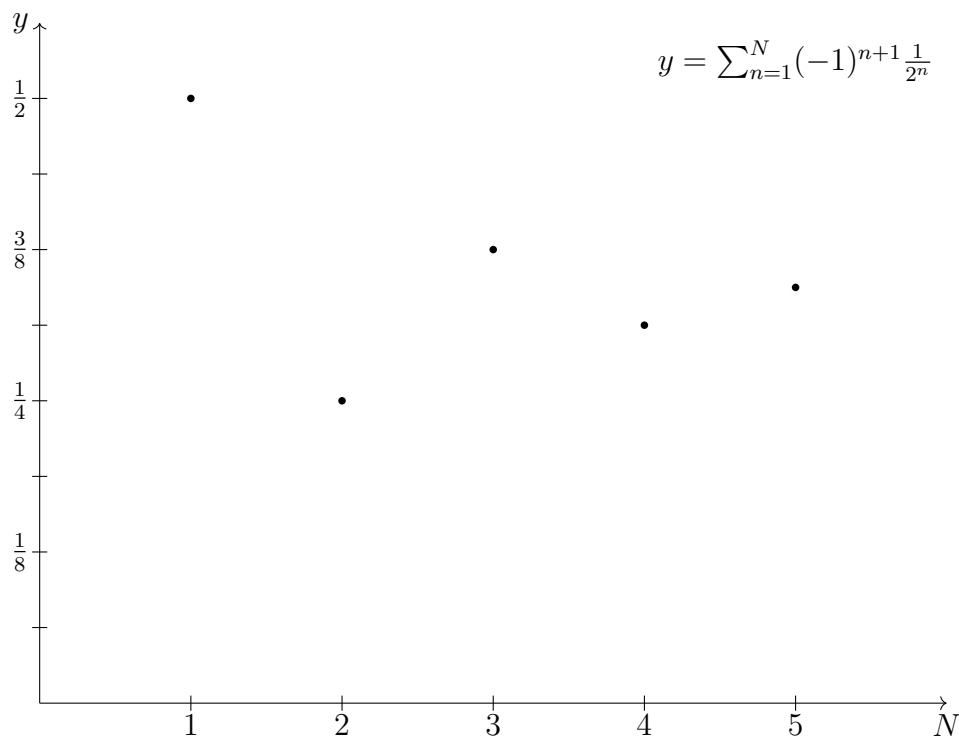


Alternating Series Test (AST): A series of the form $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converges if all of the following conditions hold:

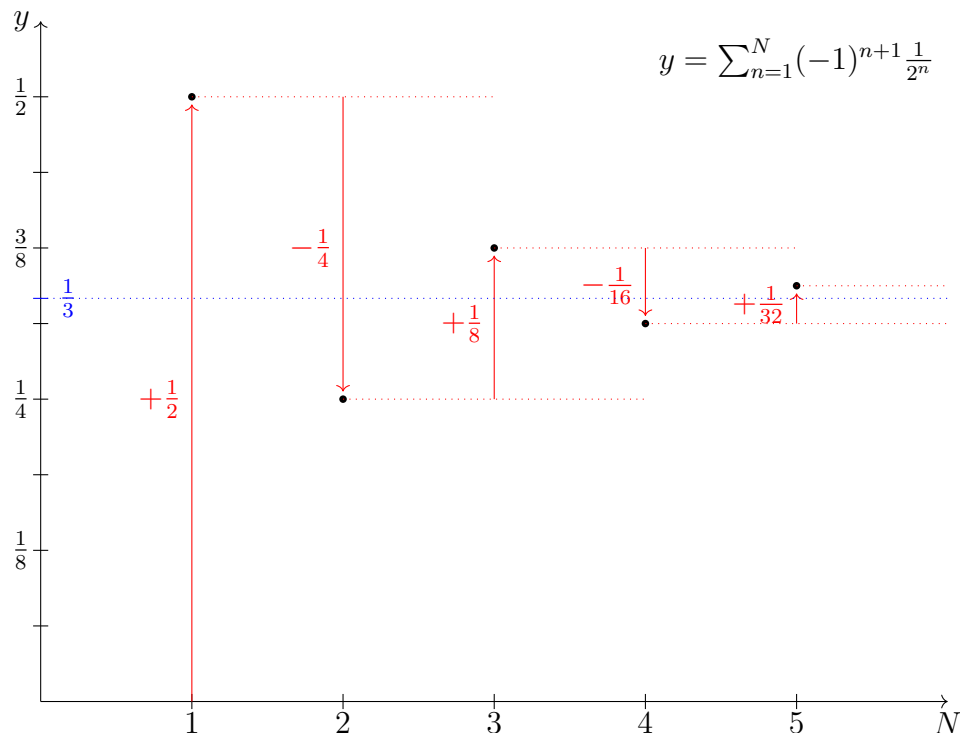
1. For all n , $b_n > 0$.
2. For all n , $|b_{n+1}| \leq |b_n|$.
3. $\lim_{n \rightarrow \infty} b_n = 0$.

In this project, you will develop an understanding of the AST by examining several series of the form $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$. Some of these series will satisfy all of the conditions in the AST, but some of these series will not.

Recall that one way to visualize a series is by plotting its partial sums. For example, a plot of the first five partial sums of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2^n}$ is given below:



This is a useful visualization, but it can be improved by adding more information. In the following plot, we have added red arrows and dotted lines to better indicate the values added in each partial sum. The limit of this series is indicated with a blue dotted line.

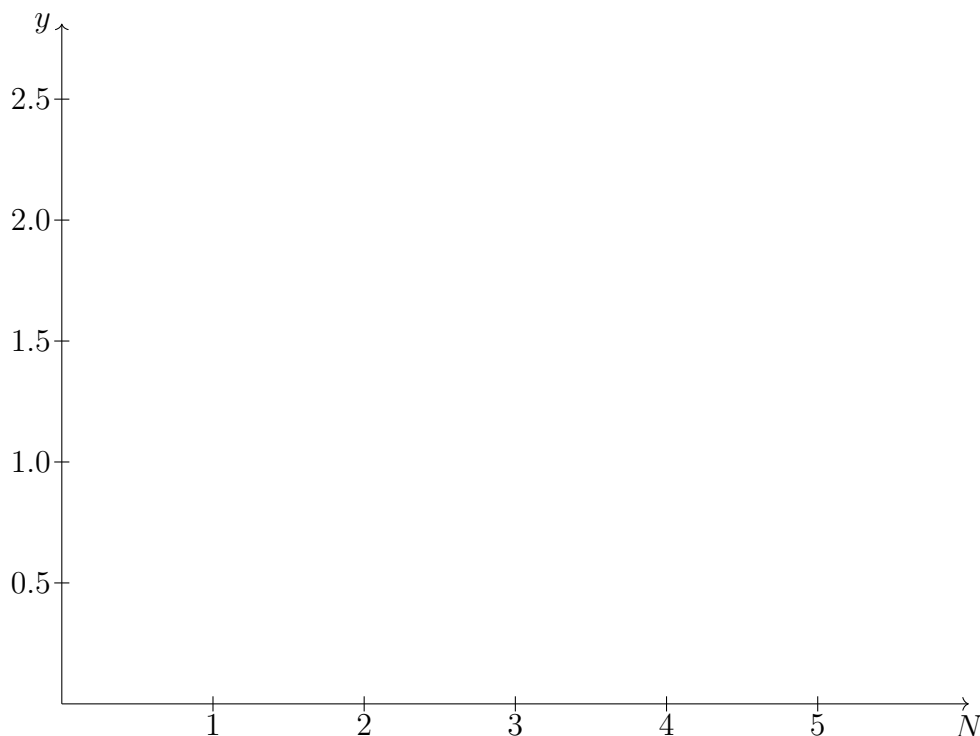


1. (a) The red arrows in this plot decrease in length. What condition of the AST does this correspond to?

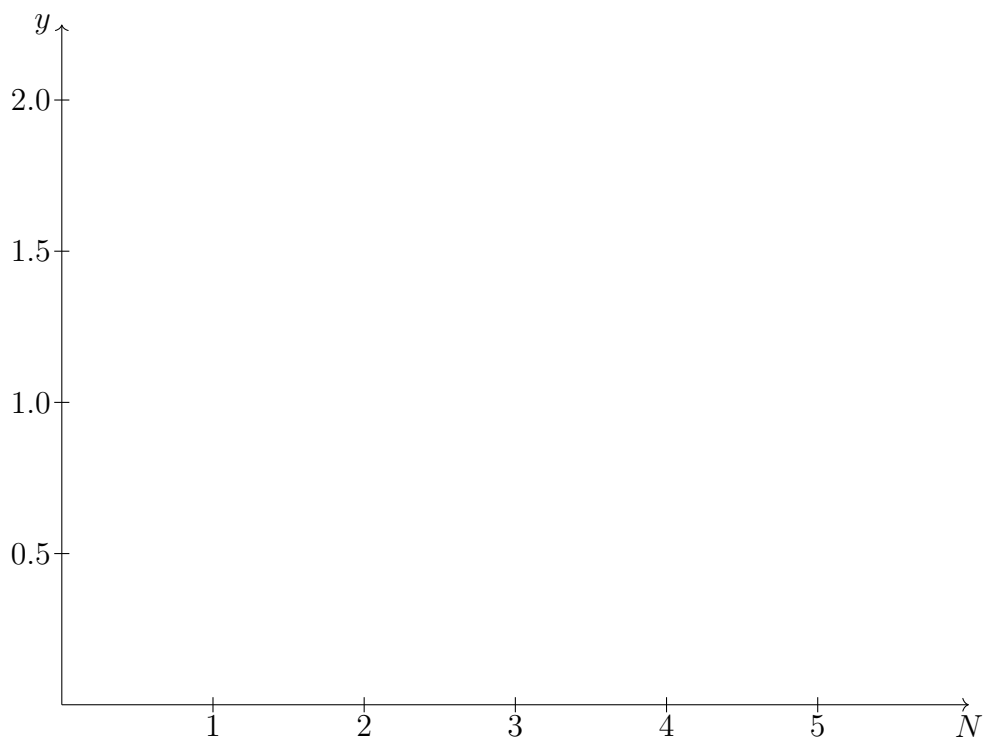
- (b) The red arrows in this plot alternate direction. What condition of the AST does this correspond to?

- (c) The red arrows in this plot have length approaching zero. What condition of the AST does this correspond to?

- (d) Each partial sum of this series is between the previous two partial sums. Do you think this happens no matter what series is plotted? If not, can you think of a combination of conditions in the AST that will force this to happen?
- (e) The values of the fourth and fifth partial sums of this series are 0.3125 and 0.34375 respectively. Can you come up with an upper and lower bound for the sixth partial sum of this series?
- (f) Can you come up with an upper and lower bound for the one hundredth partial sum of this series?
2. Plot the first five partial sums of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-1)^{n+1}}{n}$. Make sure to include arrows like above to indicate the changes between the partial sums.



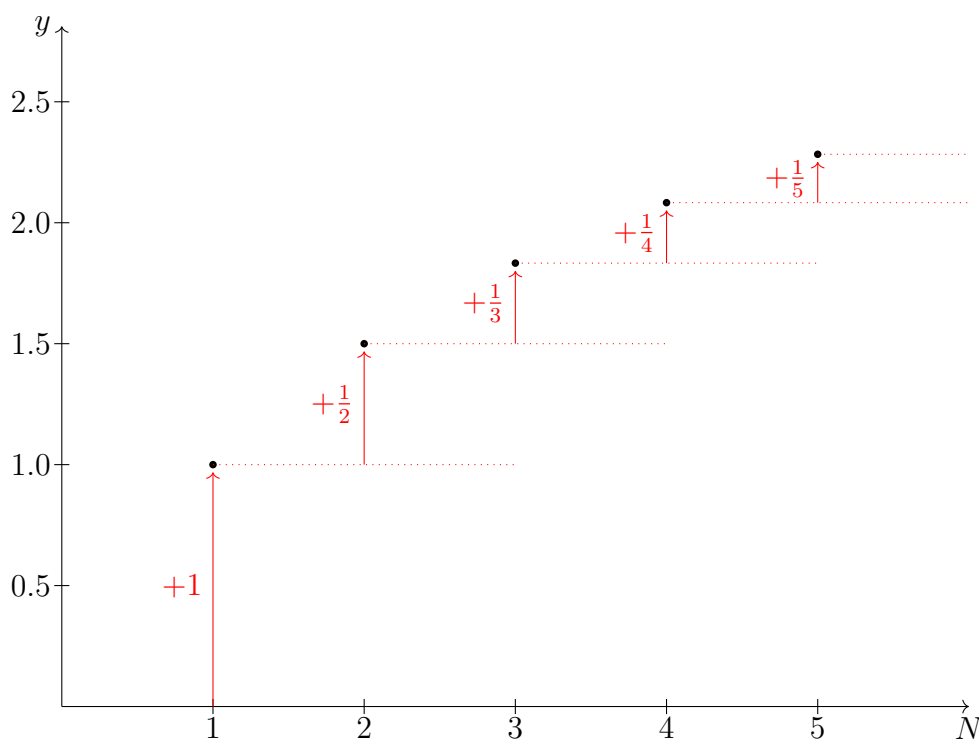
- (a) Simplify this series. Once simplified, you should recognize this series. Does it converge?
- (b) This series fails to satisfy one of the conditions of the AST. Which one? Can you describe what goes wrong in terms of the arrows between the partial sums?
3. Plot the first five partial sums of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \left(1 + \frac{1}{n}\right)$. Make sure to include arrows like above to indicate the changes between the partial sums.



(a) This series fails to satisfy one of the conditions of the AST. Which one? Can you describe what goes wrong in terms of the arrows between the partial sums?

(b) The arrows you drew should alternate direction and decrease in length. Does this force the series to converge?

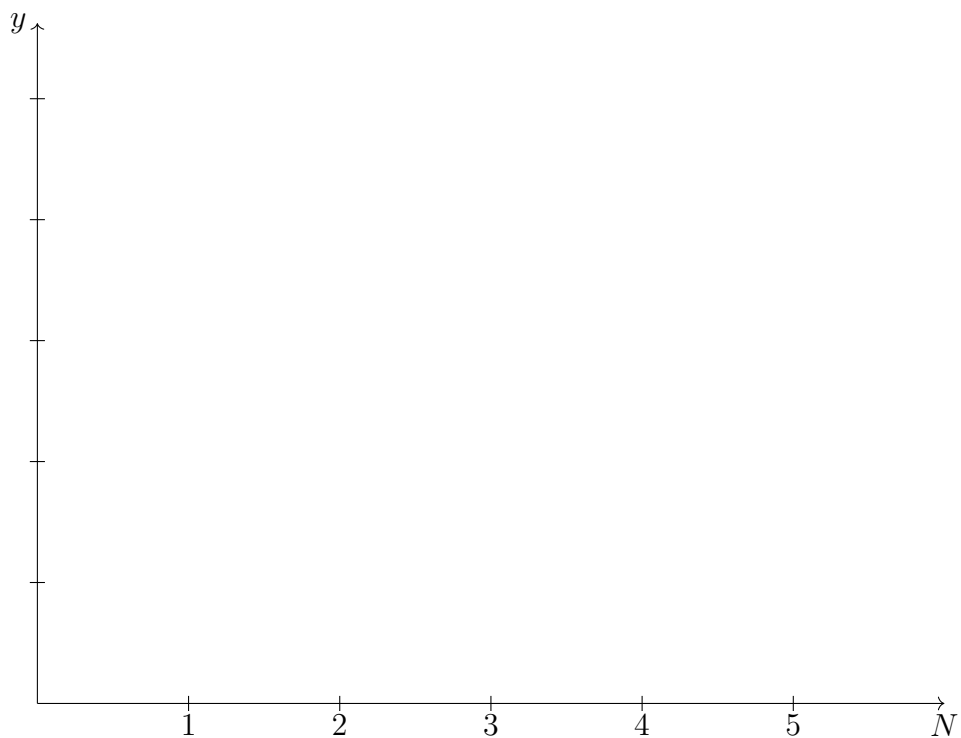
4. Consider the following plot of the first five partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{n}$ (this should be very similar to your plot in question 2):



(a) This series diverges. It is possible to create a new series from this one by “flipping arrows”. For example, we could make the second arrow point down instead of up.

Describe a way to make a convergent series from $\sum_{n=1}^{\infty} \frac{1}{n}$ just by flipping arrows.

(b) Plot the first five partial sums of the series you created:



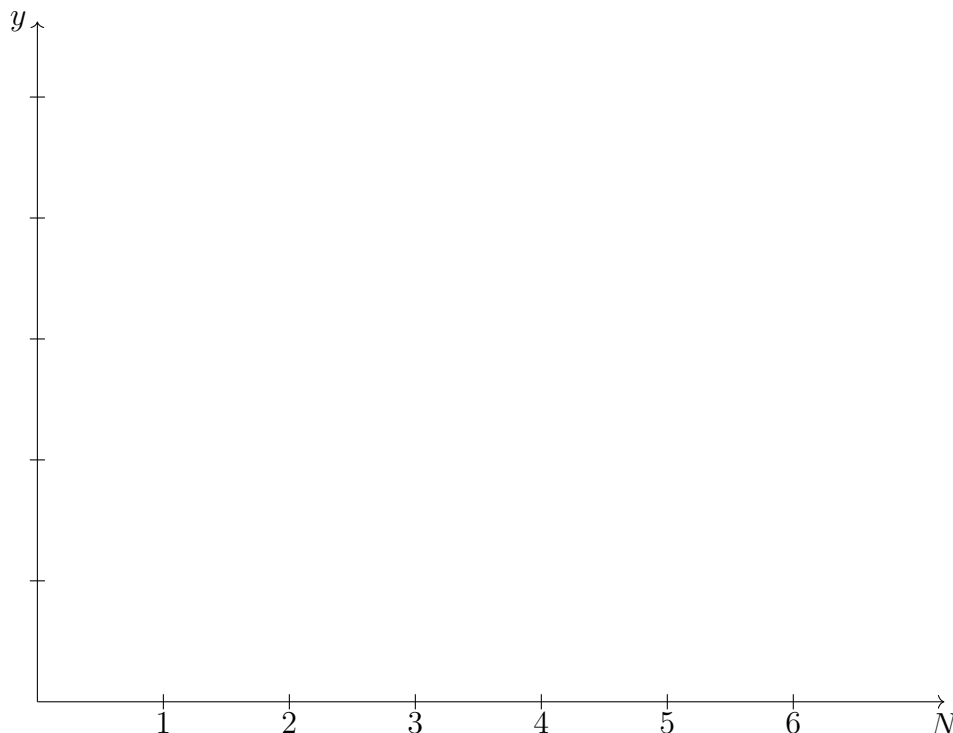
(c) Write down an algebraic expression for the series you created. (Hint: When you flip an arrow, what happens algebraically?)

5. This last example shows why we need the second condition of the AST ($|b_{n+1}| \leq |b_n|$). Consider the following definition of b_n :

$$b_n = \begin{cases} \frac{2}{n} & \text{if } n \text{ is odd} \\ \frac{1}{n-1} & \text{if } n \text{ is even} \end{cases}$$

- (a) Write down the first six values of b_n . What do you notice? This series fails to satisfy one of the conditions of the AST. Which one?

- (b) Plot the first six partial sums of the series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$.



- (c) For each N , let S_N denote the N^{th} partial sum of the series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$. Calculate $S_4 - S_2$ and $S_6 - S_4$. What do you notice? Can you write down a general formula for $S_{2N+2} - S_{2N}$?

- (d) If we define $a_n = S_{2n+2} - S_{2n}$, then it turns out that we can express even-indexed partial sums as

$$S_{2N} = S_2 + a_1 + a_2 + \cdots a_{N-2} + a_{N-1} = S_2 + \sum_{n=1}^N a_n.$$

Use your answer from part (c) to deduce that $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ diverges.