

The Integral Test

Given a series $\sum_{n=k}^{\infty} a_n$, if $f(x)$ is a function such that $f(n) = a_n$ for $n \geq k$ and:

- f is continuous
 - f is positive
 - f is decreasing
- } on $[k, \infty)$

then $\sum_{n=k}^{\infty} a_n$ converges if and only if

$\int_k^{\infty} f(x) dx$ converges.

Important:

Means if $\int_k^{\infty} f(x) dx$ converges, then can

conclude $\sum_{n=k}^{\infty} a_n$ converges

Also, if $\int_k^{\infty} f(x) dx$ diverges, then can

conclude $\sum_{n=k}^{\infty} a_n$ diverges.

example: $\sum_{n=1}^{\infty} n e^{-n^2}$ converges or diverges?

Int Test: Let $f(x) = x e^{-x^2}$. *can just state*

Check f is continuous, positive, decr.

• f is continuous $\checkmark$ b/c it's a product of continuous functions (on $[1, \infty)$) *show*

• f is positive?

$$e^{-x^2} > 0 \text{ for all } x$$

$$x > 0 \text{ if } x \geq 1$$

So f is positive. $\checkmark$

• f decreasing on $[1, \infty)$?

Show this using the derivative.

$$f'(x) = 1 \cdot e^{-x^2} + x(e^{-x^2}(-2x))$$

$$= e^{-x^2} - 2x^2 e^{-x^2}$$

$$= e^{-x^2} (1 - 2x^2)$$

$$f'(x) = e^{-x^2}(1-2x^2) = 0$$

$$e^{-x^2} = 0$$

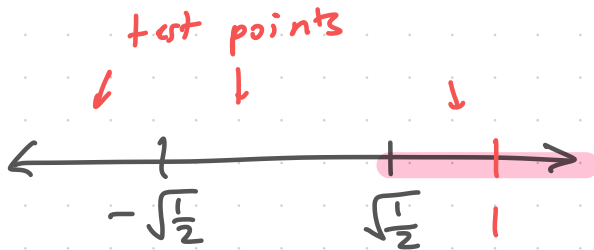
no solutions

or $1-2x^2 = 0$

$$1 = 2x^2$$

$$x^2 = \frac{1}{2}$$

$$x = \pm \sqrt{\frac{1}{2}}$$



Choose test point $x=10$: $f'(10) = \underbrace{e^{-100}}_{>0} \underbrace{(1-200)}_{-199}$

$$f'(10) < 0$$

So f is decreasing ✓ for $x > \sqrt{\frac{1}{2}}$

so decr. for $x \geq 1$.

$\Rightarrow$ We can apply the Integral Test.

$$\int_1^{\infty} x e^{-x^2} dx = \lim_{t \rightarrow \infty} \int_1^t x e^{-x^2} dx$$

$$\boxed{\begin{array}{l} u = -x^2 \\ du = -2x dx \end{array}}$$

$$\text{if } x=1, u=-1$$

$$\text{if } x=t, u=-t^2$$

$$= \lim_{t \rightarrow \infty} \int_{-1}^{-t^2} e^u \left(-\frac{1}{2}\right) du$$

$$= -\frac{1}{2} \lim_{t \rightarrow \infty} \int_{-1}^{-t^2} e^u du$$

$$= -\frac{1}{2} \lim_{t \rightarrow \infty} e^u \Big|_{-1}^{-t^2}$$

$$= -\frac{1}{2} \lim_{t \rightarrow \infty} (e^{-t^2} - e^{-1})$$

$$= -\frac{1}{2} \left(-\frac{1}{e}\right)$$

$$= \frac{1}{2e}$$

ambiguous

Converges

The improper integral converges, so the series converges by Int Test.

Note: The value of the series is not $\frac{1}{2e}$.



overestimate

ex: $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ converges or diverges?

Int Test: Let $f(x) = \frac{\ln x}{x}$.

continuous on $[2, \infty)$ ✓

positive on $[2, \infty)$ ✓

decreasing?

$$f'(x) = \frac{\left(\frac{1}{x}\right)x - \ln x(1)}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

← negative if the numerator is negative

$$1 - \ln x < 0$$

$$1 < \ln x$$

$$e < e^{\ln x}$$

$$\boxed{e < x}$$

$\Rightarrow f$ is decreasing for $x > e \approx 2.7$

↑
good enough

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{\ln n}{n} &= \frac{\ln 2}{2} + \underbrace{\frac{\ln 3}{3} + \frac{\ln 4}{4} + \dots}_{\text{converges/div.}} \\ &= \underbrace{\frac{\ln 2}{2}}_{\text{a number}} + \underbrace{\sum_{n=3}^{\infty} \frac{\ln n}{n}}_{\text{converges/d.v.}} \end{aligned}$$

This tells us if the "tail" of the series converges then the whole series converges.

$$\begin{aligned} \int_2^{\infty} \frac{\ln x}{x} dx &= \lim_{t \rightarrow \infty} \int_2^t \frac{\ln x}{x} dx \quad \boxed{u = \ln x, \quad du = \frac{1}{x} dx} \\ &= \lim_{t \rightarrow \infty} \int_{\ln 2}^{\ln t} u \, du \\ &= \lim_{t \rightarrow \infty} \left[\frac{1}{2} u^2 \right]_{\ln 2}^{\ln t} \\ &= \lim_{t \rightarrow \infty} \frac{1}{2} (\ln t)^2 - \frac{1}{2} (\ln 2)^2 \end{aligned}$$

$$= \infty - \frac{1}{2} (\ln 2)^2$$

$$= \infty$$

the integral diverges, so the series diverges by the Integral Test.

p-Series

A p-series is a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where p is a real number.

Recall: $\int_1^{\infty} \frac{1}{x^p} dx$ converges if $p > 1$
diverges if $p \leq 1$

$$\underbrace{\int_1^{\infty} \frac{1}{x} dx}_{p=1} \text{ diverges} \quad / \quad \int_1^{\infty} \frac{1}{x^2} dx \text{ converges}$$

p-series Test: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$
diverges if $p \leq 1$

$$\boxed{\sum \frac{1}{n} \text{ harmonic, diverges} / \sum \frac{1}{n^2} \text{ converges}}$$

$$\text{ex: } \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$$

$$p = \frac{1}{3} < 1$$

so the series
diverges by
p-test

$$\begin{aligned} \text{ex: } \sum_{n=1}^{\infty} \frac{1}{n^{5/2}} &= \sum_{n=1}^{\infty} \frac{n^{1/2}}{n^2} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{2-1/2}} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \end{aligned}$$

$$p = \frac{3}{2} > 1$$

so series
converges.