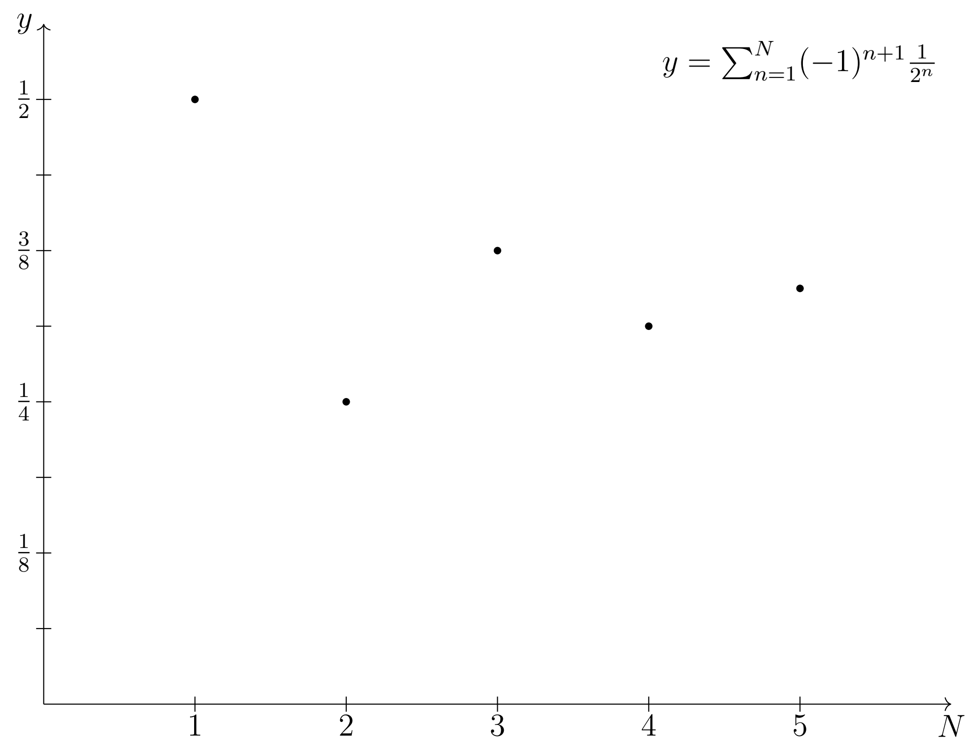
Alternating Series Test (AST): A series of the form $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ converges if all of the following conditions hold:

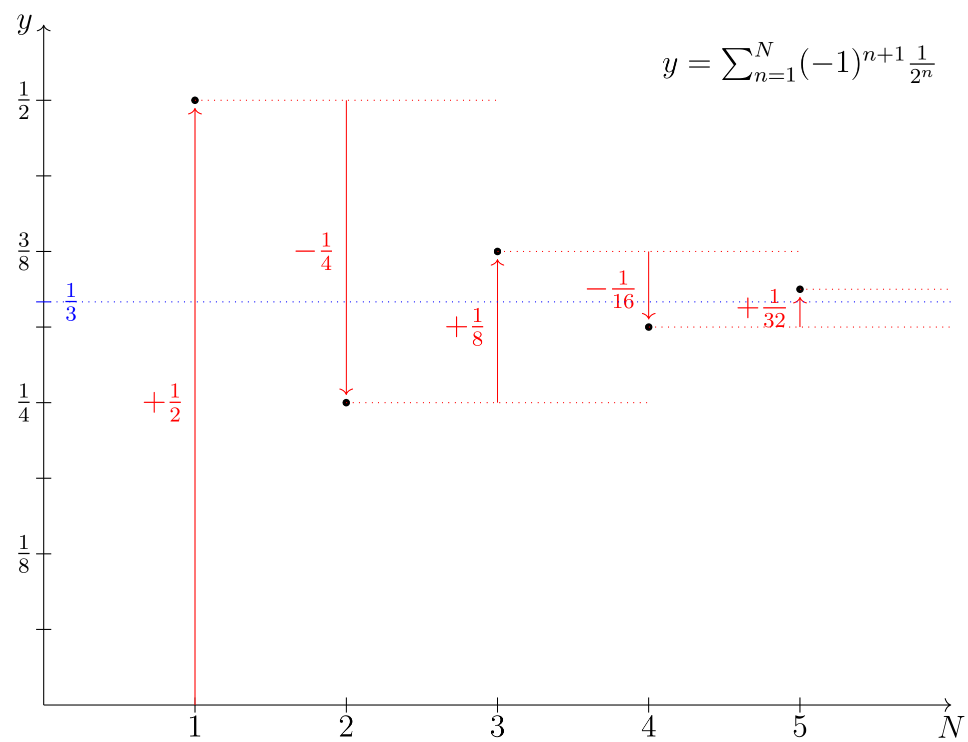
1. For all n, $b_n > 0$.
2. For all n, $|b_{n+1}| \le |b_n|$.
3. $\lim_{n\to\infty} b_n = 0$.

In this project, you will develop an understanding of the AST by examining several series of the form $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$. Some of these series will satisfy all of the conditions in the AST, but some of these series will not.

Recall that one way to visualize a series is by plotting its partial sums. For example, a plot of the first five partial sums of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2^n}$ is given below:



This is a useful visualization, but it can be improved by adding more information. In the following plot, we have added red arrows and dotted lines to better indicate the values added in each partial sum. The limit of this series is indicated with a blue dotted line.



1. (a) The red arrows in this plot decrease in length. What condition of the AST does this correspond to?

   This corresponds to condition 2. $|b_n|$ is the length of the n^{th} arrow, so "For all n, $|b_{n+1}| \leq |b_n|$" means "For all n, the length of arrow $n+1$ is less than or equal to the length of arrow n".

   (b) The red arrows in this plot alternate direction. What condition of the AST does this correspond to?

   This corresponds to condition 1. The direction of arrow n depends on the sign of $(-1)^{n+1}b_n$. If each b_n is positive, then $(-1)^{n+1}b_n$ alternates between being positve and negative.

   (c) The red arrows in this plot have length approaching zero. What condition of the AST does this correspond to?

   This corresponds to condition 3. If $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} |b_n| = 0$ as well.

(d) Each partial sum of this series is between the previous two partial sums. Do you think this happens no matter what series is plotted? If not, can you think of a combination of conditions in the AST that will force this to happen?

This does not always happen. For example, see problem 2. Conditions 1 and 2 of the AST force each partial sum to be between the previous two.

Suppose S_n, S_{n+1} and S_{n+2} are three consecutive partial sums. If $S_n \geq S_{n+1}$, then the arrow connecting S_n to S_{n+1} points downwards. Condition 1 forces the next arrow to point upwards, so S_{n+1} will be below S_{n+2}. Condition 2 of the AST forces the arrow from S_n to S_{n+1} to be longer than the arrow from S_{n+1} to S_{n+2}, so S_{n+2} will be below S_n. Overall, we have $S_{n+1} \leq S_{n+2} \leq S_n$.

A similar argument can be made when the arrow connecting S_n to S_{n+1} points upwards.

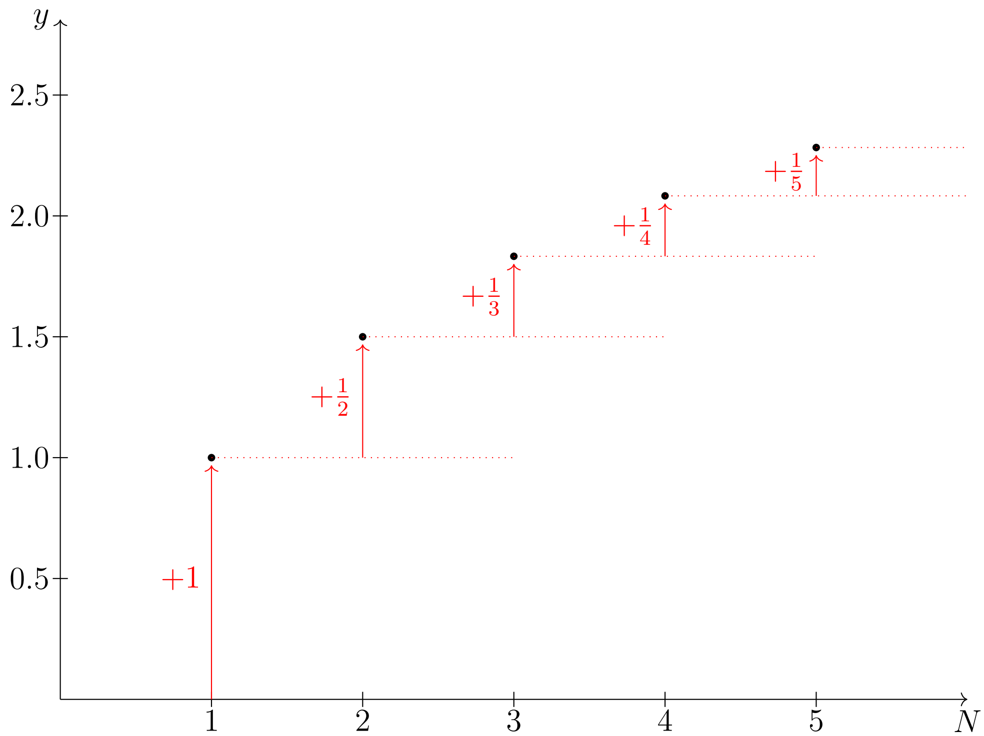
(e) The values of the fourth and fifth partial sums of this series are 0.3125 and 0.34375 respectively. Can you come up with an upper and lower bound for the sixth partial sum of this series?

Each partial sum lies between the previous two, so the sixth partial sum is between 0.3125 and 0.34375.

(f) Can you come up with an upper and lower bound for the one hundredth partial sum of this series?

Yes. We can extend our argument in part (d) to show that each partial sum lies between any pair of earlier consecutive partial sums. In particular, the one hundredth partial sum lies between the fourth and fifth partial sums, so the one hundredth partial sum is also between 0.3125 and 0.34375.

2. Plot the first five partial sums of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-1)^{n+1}}{n}$. Make sure to include arrows like above to indicate the changes between the partial sums.



(a) Simplify this series. Once simplified, you should recognize this series. Does it converge?

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty}\frac{\left((-1)^{n+1}\right)^2}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$$

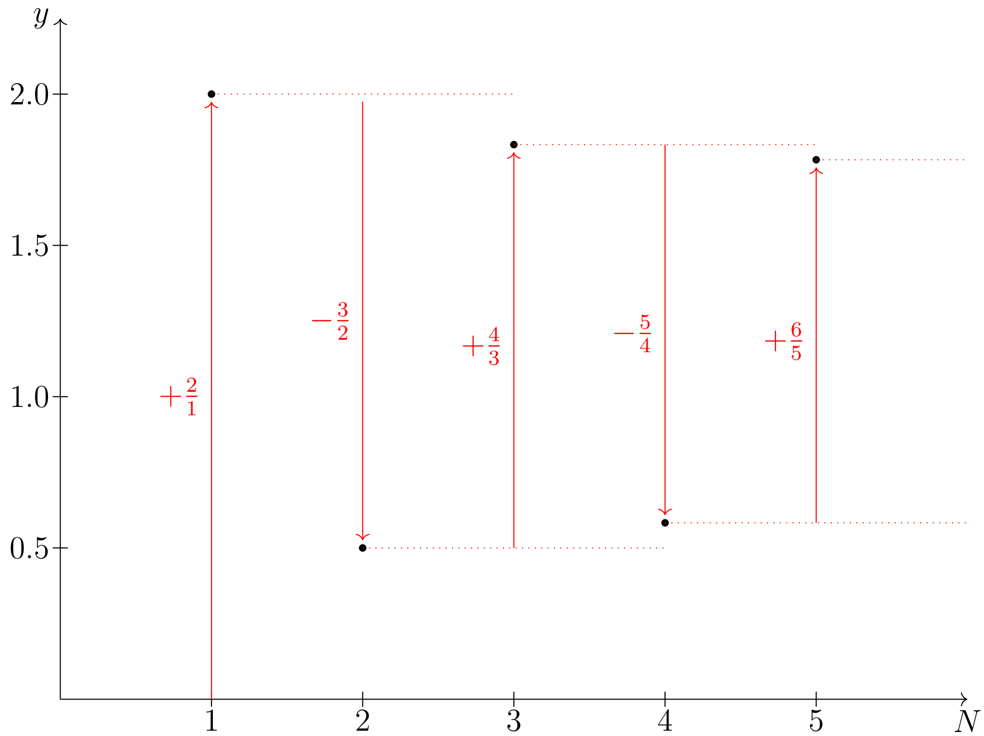
To see that the second to last equality is true, note that $(-1)^{n+1}$ is either 1 or -1, depending on n. In either case, the square of this value is 1.

This is the harmonic series. It does not converge.

(b) This series fails to satisfy one of the conditions of the AST. Which one? Can you describe what goes wrong in terms of the arrows between the partial sums?

This series fails to satisfy condition 1 of the AST because $b_n = \frac{(-1)^{n+1}}{n}$ is not always positive. We can see that this series fails condition 1 of the AST since the arrows in the above plot do not alternate direction.

3. Plot the first five partial sums of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\left(1+\frac{1}{n}\right)$. Make sure to include arrows like above to indicate the changes between the partial sums.



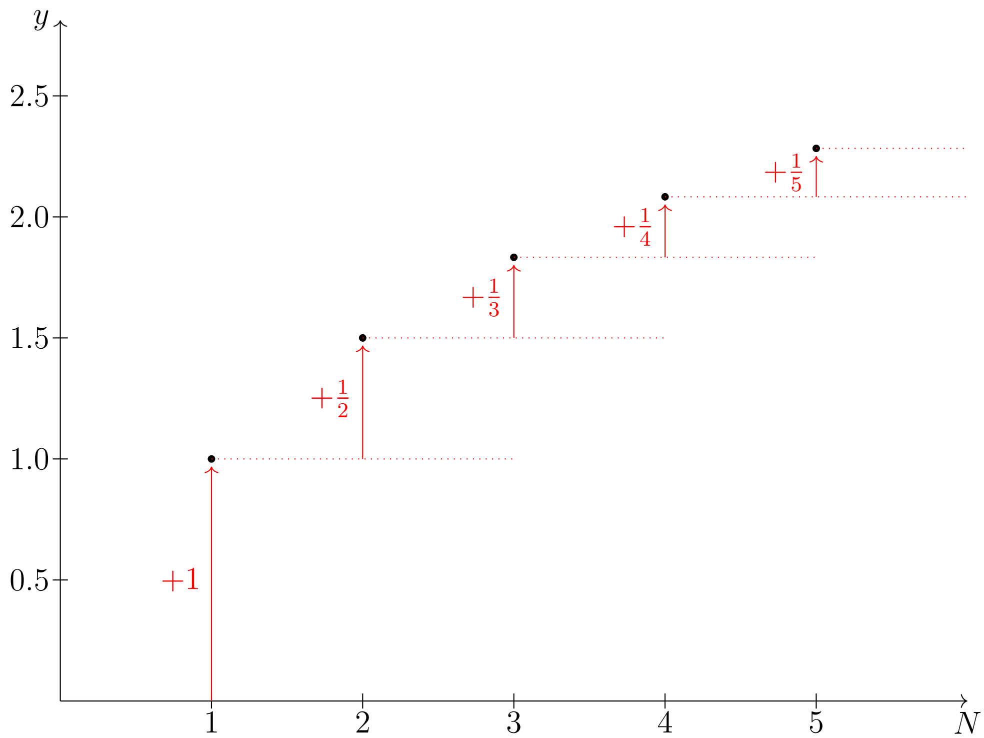
(a) This series fails to satisfy one of the conditions of the AST. Which one? Can you describe what goes wrong in terms of the arrows between the partial sums?

This series fails condition 3 of the AST. The lengths of arrows approach 1, but condition 3 requires the lengths to approach 0.

(b) The arrows you drew should alternate direction and decrease in length. Does this force the series to converge?

No, this is not enough. In fact, we can see by using the Divergence Test that this series diverges. Specifically, since $\lim_{n\to\infty}(-1)^{n+1}\left(1+\frac{1}{n}\right)$ does not exist, the series diverges.

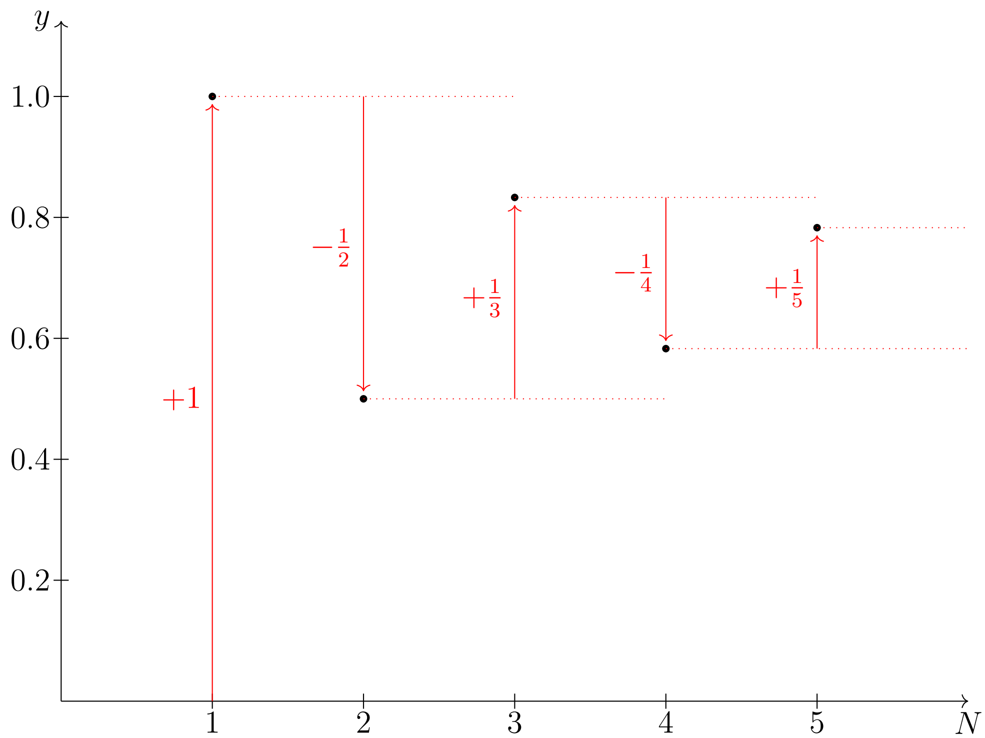
4. Consider the following plot of the first five partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{n}$ (this should be very similar to your plot in question 2):



(a) This series diverges. It is possible to create a new series from this one by "flipping arrows". For example, we could make the second arrow point down instead of up. Describe a way to make a convergent series from $\sum_{n=1}^{\infty} \frac{1}{n}$ just by flipping arrows.

This series already satisfies conditions 2 and 3 of the AST, and flipping arrows won't change that. For this reason, we just need to ensure that arrows alternate in direction. To accomplish this, we can flip every other arrow / every arrow with even index.

(b) Plot the first five partial sums of the series you created:



(c) Write down an algebraic expression for the series you created. (Hint: When you flip an arrow, what happens algebraically?)

Flipping an arrow corresponds to multiplying by -1. We want to multiply every other term of the underlying sequence by -1, so we can multiply by $(-1)^{n+1}$ (this equals 1 when n is odd and -1 when n is even). The final expression is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

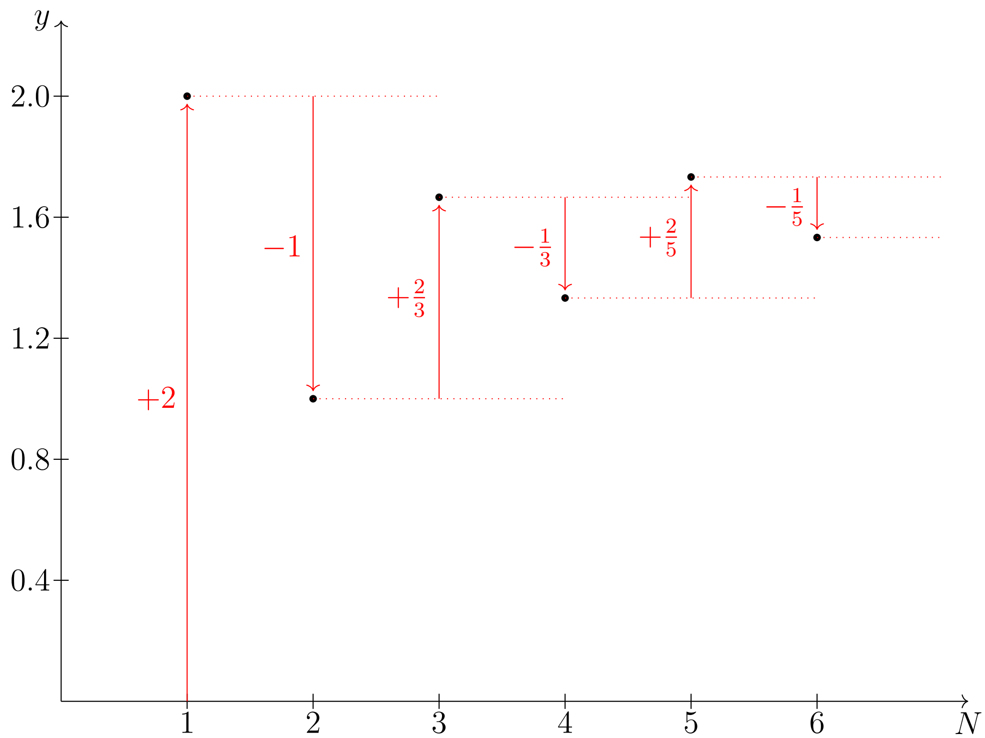
5. This last example shows why we need the second condition of the AST ($|b_{n+1}| \leq |b_n|$). Consider the following definition of b_n:

$$b_n = \begin{cases} \frac{2}{n} & \text{if } n \text{ is odd} \\ \frac{1}{n-1} & \text{if } n \text{ is even} \end{cases}$$

(a) Write down the first six values of b_n. What do you notice? This series fails to satisfy one of the conditions of the AST. Which one?

The first six values of b_n are 2, -1, $\frac{2}{3}$, $-\frac{1}{3}$, $\frac{2}{5}$, and $-\frac{1}{5}$. We can describe a pattern in these terms by grouping them into pairs (e.g., 2 with -1). It appears that the first item in each pair is twice as large and has opposite sign as the second item in each pair.
This series fails the second condition of the AST. For example, $|b_4| = \frac{1}{3} \approx 0.333$ is smaller than $|b_5| = \frac{2}{5} = 0.4$.

(b) Plot the first six partial sums of the series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$.



(c) For each N, let S_N denote the N^{th} partial sum of the series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$. Calculate $S_4 - S_2$ and $S_6 - S_4$. What do you notice? Can you write down a general formula for $S_{2N+2} - S_{2N}$?

$S_4 - S_2 = b_4 + b_3 = -\frac{1}{3} + \frac{2}{3} = \frac{1}{3}$ and $S_6 - S_4 = b_6 + b_5 = -\frac{1}{5} + \frac{2}{5} = \frac{1}{5}$.
In general, $S_{2N+2} - S_{2N} = b_{2N+2} + b_{2N+1} = -\frac{1}{2N+1} + \frac{2}{2N+1} = \frac{1}{2N+1}$.

(d) If we define $a_n = S_{2n+2} - S_{2n}$, then it turns out that we can express even-indexed partial sums as

$$S_{2N} = S_2 + a_1 + a_2 + \cdots a_{N-2} + a_{N-1} = S_2 + \sum_{n=1}^{N} a_n.$$

Use your answer from part (c) to deduce that $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ diverges.

From part (c), we have

$$S_{2N} = S_2 + \sum_{n=1}^{N} a_n = S_2 + \sum_{n=1}^{N} S_{2n+2} - S_{2n} = S_2 + \sum_{n=1}^{N} \frac{1}{2n+1}$$

We can manipulate the righthand sum into one more recognizable. Note that for each $n \geq 1$, we have $\frac{1}{2n+1} \geq \frac{1}{2n+2n} = \frac{1}{4n}$, so we actually have:

$$S_{2N} = S_2 + \sum_{n=1}^{N} \frac{1}{2n+1} \geq S_2 + \sum_{n=1}^{N} \frac{1}{4n} = S_2 + \frac{1}{4}\sum_{n=1}^{N} \frac{1}{n}$$

However, we know

$$\lim_{N\to\infty}\left[S_2+\frac{1}{4}\sum_{n=1}^{N}\frac{1}{n}\right]=\infty$$

(note the presence of the harmonic series!) Hence, we also have

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n=\lim_{N\to\infty}S_{2N}=\infty$$