

Trig Substitution

An integration technique for integrals that

contain: $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$, $\sqrt{x^2 + a^2}$

Where a is a constant.

E.g. $\int \frac{1}{x^2 \sqrt{9 - x^2}} dx$ or $\int \sqrt{x^2 + 1} dx$

ex: $\int \frac{1}{x^2 \sqrt{9 - x^2}} dx$

old variable (x)
new variable (θ)

Use the substitution $x = 3 \sin \theta$

Why? It simplifies the square root.

$$\sqrt{9 - x^2} = \sqrt{9 - (3 \sin \theta)^2}$$

$$= \sqrt{9 - 9 \sin^2 \theta}$$

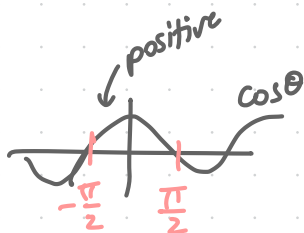
$$= \sqrt{9 (1 - \sin^2 \theta)}$$

$$= \sqrt{9} \sqrt{\cos^2 \theta}$$

$$= 3 |\cos \theta|$$

$$= 3 \cos \theta \quad \text{if } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$\sqrt{x^2} = |x|$$



$$\int \frac{1}{x^2 \sqrt{9-x^2}} dx$$

$$x = 3 \sin \theta$$

$$dx = 3 \cos \theta d\theta$$

$$= \int \frac{\cancel{3 \cos \theta}}{(3 \sin \theta)^2 \cdot \cancel{3 \cos \theta}} d\theta$$

$$= \int \frac{1}{9 \sin^2 \theta} d\theta$$

$$= \frac{1}{9} \int \csc^2 \theta d\theta$$

$$= \frac{1}{9} (-\cot \theta) + C$$

$$= -\frac{1}{9} \cot \theta + C$$

Need to sub back to x : $x = 3 \sin \theta$

$$\frac{x}{3} = \sin \theta$$

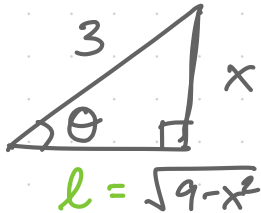
$$\arcsin\left(\frac{x}{3}\right) = \theta$$

$$= -\frac{1}{9} \cot\left(\arcsin\left(\frac{x}{3}\right)\right) + C$$

trig (inverse trig (?))

can be simplified

We have that $\sin \theta = \frac{x}{3} = \frac{\text{opposite}}{\text{hypotenuse}}$



$$x^2 + l^2 = 9$$

$$l^2 = 9 - x^2$$

$$l = \pm \sqrt{9 - x^2}$$

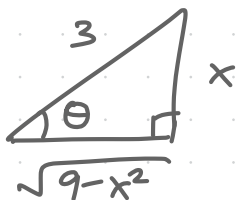
$$\Rightarrow \cot \theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{\sqrt{9 - x^2}}{x}$$

Answer: $-\frac{1}{9} \cot \theta + C = -\frac{\sqrt{9 - x^2}}{9x} + C$

~~$\int dx \rightarrow \int d\theta$~~

~~$= \frac{1}{2} \theta + \sin \theta + C$~~

$\uparrow$
 $\arcsin\left(\frac{x}{3}\right)$



represents the substitution
 $x = 3 \sin \theta$.

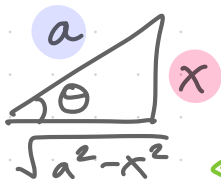
Notice $\frac{\sqrt{9-x^2}}{3} = \frac{\text{adj.}}{\text{hyp.}} = \cos \theta$

$$\sqrt{9-x^2} = 3 \cos \theta$$

Recall: the 3 possibilities

$$\sqrt{a^2-x^2}, \sqrt{x^2-a^2}, \sqrt{x^2+a^2}$$

- If you see $\sqrt{a^2-x^2}$, use $x = a \sin \theta$
 since $\sin \theta = \frac{x}{a} = \frac{\text{opp}}{\text{hyp}}$



← comes from the Pythagorean Thm.

Then from the triangle, $\frac{\sqrt{a^2-x^2}}{a} = \cos \theta$

$$\sqrt{a^2-x^2} = a \cos \theta$$

use for replacing $\sqrt{a^2-x^2}$

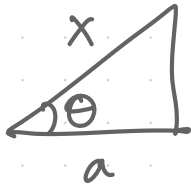
E.g. for first example

$$x = 3 \sin \theta \implies dx = 3 \cos \theta d\theta$$

and $3 \cos \theta = \sqrt{9 - x^2}$ (from triangle or by simplifying)

$$\int \frac{dx}{x^2 \sqrt{9 - x^2}} = \int \frac{3 \cos \theta d\theta}{(3 \sin \theta)^2 3 \cos \theta}$$

- If you see $\sqrt{x^2 - a^2}$, use $x = a \sec \theta$
 Since $\sec \theta = \frac{x}{a} = \frac{\text{hyp}}{\text{adj}}$



$$l = \sqrt{x^2 - a^2}$$

Pythag. Thm. $\rightarrow$

$$a^2 + l^2 = x^2$$

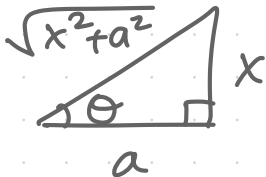
$$l^2 = x^2 - a^2$$

$$l = \pm \sqrt{x^2 - a^2}$$

From the triangle: $\frac{\sqrt{x^2 - a^2}}{a} = \frac{\text{opp}}{\text{adj}} = \tan \theta$

$$\sqrt{x^2 - a^2} = a \tan \theta$$

- If you see $\sqrt{x^2 + a^2}$, use $x = a \tan \theta$
 Since $\tan \theta = \frac{x}{a} = \frac{\text{opp}}{\text{adj}}$



From the triangle: $\frac{\sqrt{x^2 + a^2}}{a} = \frac{\text{hyp}}{\text{adj}} = \sec \theta$

$$\sqrt{x^2 + a^2} = a \sec \theta$$

ex: $\int \frac{1}{x^2 \sqrt{x^2 - 25}} dx$

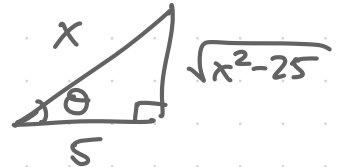
$a=5$

Since we see $\sqrt{x^2 - 25}$, we $x = 5 \sec \theta$

$$\boxed{\begin{aligned} x &= 5 \sec \theta \\ dx &= 5 \sec \theta \tan \theta d\theta \end{aligned}}$$

$$\sec \theta = \frac{x}{5} = \frac{\text{hyp}}{\text{adj.}}$$

so



From the triangle,

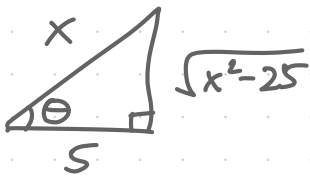
$$\frac{\sqrt{x^2 - 25}}{5} = \frac{\text{opp}}{\text{adj.}} = \tan \theta$$

$$\sqrt{x^2 - 25} = 5 \tan \theta$$

$$\begin{aligned} \int \frac{1}{x^2 \sqrt{x^2 - 25}} dx &= \int \frac{\cancel{5 \sec \theta \tan \theta}}{(5 \sec \theta)^2 \cdot \cancel{5 \tan \theta}} d\theta \\ &= \int \frac{\cancel{\sec \theta}}{25 \sec^2 \theta} d\theta \end{aligned}$$

$$\begin{aligned}
 &= \int \frac{1}{25 \sec \theta} d\theta \\
 &= \frac{1}{25} \int \frac{1}{\sec \theta} d\theta \\
 &= \frac{1}{25} \int \cos \theta d\theta \\
 &= \frac{1}{25} \sin \theta + C
 \end{aligned}$$

To sub back, use the triangle:



$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{x^2 - 25}}{x}$$

Answer: $\frac{1}{25} \cdot \frac{\sqrt{x^2 - 25}}{x} + C$

$$\begin{aligned}
 x &= 5 \sec \theta \\
 \theta &= \operatorname{arccsc}\left(\frac{x}{5}\right)
 \end{aligned}$$

Standard

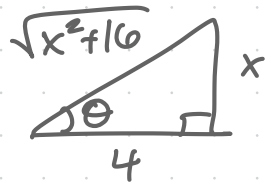
subs

$$x = a \sin \theta \quad / \quad x = a \sec \theta \quad / \quad x = a \tan \theta$$

example: $\int \frac{\sqrt{x^2+16}}{x^4} dx$

from Pythag.
Thm.

Use $x = 4 \tan \theta$
 $dx = 4 \sec^2 \theta d\theta$



$$\tan \theta = \frac{x}{4} = \frac{\text{opp}}{\text{adj.}}$$

From triangle,

What to replace
 $\sqrt{x^2+16}$ with
in terms of θ

$$\left\{ \frac{\sqrt{x^2+16}}{4} = \frac{\text{hyp}}{\text{adj.}} = \sec \theta \right.$$

$$\sqrt{x^2+16} = 4 \sec \theta$$

$$\int \frac{\sqrt{x^2+16}}{x^4} dx = \int \frac{4 \sec \theta}{(4 \tan \theta)^4} \cdot 4 \sec^2 \theta d\theta$$

$$= \int \frac{4^2 \sec^3 \theta}{4^4 \tan^4 \theta} d\theta$$

$$= \frac{1}{4^2} \int \frac{\sec^3 \theta}{\tan^4 \theta} d\theta$$

← write in
terms
of sin/cos

$$= \frac{1}{16} \int \frac{\cos^3 \theta}{\sin^4 \theta \cos^4 \theta} d\theta$$

$$= \frac{1}{16} \int \frac{1}{\cancel{\cos^2 \theta}} \cdot \frac{\cos^1 \theta}{\sin^4 \theta} d\theta$$

$$= \frac{1}{16} \int \frac{\cos \theta}{\sin^4 \theta} d\theta \quad du$$

$$\boxed{\begin{array}{l} u = \sin \theta \\ du = \cos \theta d\theta \end{array}}$$

$$= \frac{1}{16} \int \frac{1}{u^4} du$$

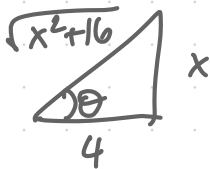
$$= \frac{1}{16} \int u^{-4} du$$

$$= \frac{1}{16} \left(-\frac{1}{3} u^{-3} \right) + C$$

$$= -\frac{1}{48} \cdot \frac{1}{u^3} + C$$

$$= -\frac{1}{48} \cdot \frac{1}{\sin^3 \theta} + C$$

$$= -\frac{1}{48} \left(\frac{1}{\sin \theta} \right)^3 + C$$



$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{x}{\sqrt{x^2 + 16}} \Rightarrow \frac{1}{\sin \theta} = \frac{\sqrt{x^2 + 16}}{x}$$

Answer: $-\frac{1}{48} \left(\frac{\sqrt{x^2+16}}{x} \right)^3 + C$

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \\ \tan^2 \theta + 1 &= \sec^2 \theta \end{aligned} \quad \left. \vphantom{\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \\ \tan^2 \theta + 1 &= \sec^2 \theta \end{aligned}} \right\} \text{ useful!}$$

u-sub with $u = \sin \theta / \cos \theta / \sec \theta / \tan \theta$

example: definite integral $\Rightarrow$ change the bounds

$$\int_0^2 \frac{x^3}{\sqrt{x^2+4}} dx$$

$$x = 2 \tan \theta$$

$$dx = 2 \sec^2 \theta d\theta$$



from triangle,

$$\frac{\sqrt{x^2+4}}{2} = \frac{\text{hyp}}{\text{adj}} = \sec \theta$$

$$\sqrt{x^2+4} = 2 \sec \theta$$

change bounds: using $x = 2 \tan \theta$

$$\text{if } x = 0 \Rightarrow 0 = 2 \tan \theta$$

$$\tan \theta = 0$$

$$\boxed{\theta = 0}$$

← domain restriction comes in here...

$$\text{if } x = 2 \Rightarrow 2 = 2 \tan \theta$$

$$\tan \theta = 1$$

$$\boxed{\theta = \pi/4}$$

$$\int_0^2 \frac{x^3}{\sqrt{x^2+4}} dx = \int_0^{\pi/4} \frac{(2 \tan \theta)^3}{\cancel{2 \sec \theta}} \cdot \cancel{2 \sec^2 \theta} d\theta$$

$$= \int_0^{\pi/4} 2^3 \tan^3 \theta \cdot \sec \theta d\theta$$

$$= 8 \int_0^{\pi/4} \tan^3 \theta \sec \theta d\theta$$

$$u = \sec \theta \\ du = \sec \theta \tan \theta d\theta$$

OR

$$u = \tan \theta \\ du = \sec^2 \theta d\theta$$

to be continued...