STOCHASTIC CONTROL SYSTEMS WITH AN EXPONENTIAL-OF-INTEGRAL PERFORMANCE INDEX

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Abstract. In this project, the optimal control of linear stochastic systems with an exponential of the integral performance index is discussed for both the full and partial observation cases. A complete solution to the optimal control problem (under quadratic payoff) with partial information using the Girsanov transformation approach is illustrated. This is then applied to the partial observation case where the pay-off is given as an exponential of the standard quadratic functional.

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1. Introduction

Stochastic control theory plays a vital role in decision-making under uncertainty in numerous systems that arise in the fields of economics, finance, biology, and engineering. This project aims to study the theoretical foundations and associated analysis of stochastic controls, numerical methods for solving stochastic control problems, and their applications.

1.1. Preliminaries. A typical non-linear stochastic control problem looks like

$$\begin{aligned} dX_s &= b(X_s, u(s))\,ds + \sigma(X_s)\,dW_s, \quad 0 \le s \le T, \\ X_0 &= x^0 \in \mathbb{R}^n, \end{aligned} \tag{1.1}$$

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where

- X_s is the state at time s,
- $u(s)$ is the m-dimensional input at time s or the control,
- $\{W_s, s \geq 0\}$ be a l-dimensional Brownian motion on $(\Omega, \mathcal{A}, P)$.
- Infinitesimal generator of the (1.1) is given as

$$\mathscr{L}^u f(z) = \frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2 f}{\partial z_i \partial z_j}(x) + \sum_i b_i(x,u)\frac{\partial f}{\partial z_i}(x), \quad x \in \mathbb{R}^n \tag{1.2}$$

  where $a_{i,j}(z) \stackrel{\text{def}}{=} \big(\sigma(x)\sigma^T(z)\big)_{ij}$ and the superscript u denotes that the generator depends on the control u.

For $\varphi \in C_b^2(\mathbb{R}^n, \mathbb{R})$, the Itô lemma gives

$$\begin{aligned}\varphi(X_t, t) = \varphi(x^0) &+ \int_0^t \{\partial_s \varphi(X_s, s) + \mathscr{L}^u \varphi(X_s, s)\} ds \\ &+ \int_0^t \nabla \varphi^T(X_s, s)\sigma(X_s, s) dW_s. \end{aligned} \tag{1.3}$$

One of the method used to solve stochastic control problems is dynamic programing techniques [8, 9], which can be summed as:

Admissible control $u^*(\cdot) \in \mathcal{U}$ is the optimal control for given stochastic control problems under suitable cost functional $J(\cdot)$ in the time interval $[0, T]$, if

$$J[u^*(\cdot)] \leq J[u(\cdot)], \quad \forall u(\cdot) \in \mathcal{U}.$$

1.2. Objective. In this project, we restrict ourselves to linear dynamics. Classical stochastic control with partial observation is well treated in the literature (see [1, 2, 3] and references therein). In most cases, optimal control is often derived via the *separation principle*, where a feedback rule based on the Kalman filter (i.e., conditional mean) is used (see [3, Chapters 1 and 2]). This assumes the existence of sufficient statistics that are *finite-dimensional*, typically the conditional mean. Generally, so-called *sufficient statistics* are infinite-dimensional (e.g., the full conditional probability distribution), and the separation principle fails. This project will be primarily based on considering the case where we have

- The sufficient statistics are finite-dimensional (same as the state space) but do not coincide with the conditional mean.
- The optimal control is given by feedback on these statistics.
- The system has linear dynamics and observation, with a cost functional (pay-off) that is exponential-quadratic, modeling *risk sensitivity*.

We will first solve the full-observation case in Section 2, then we briefly demonstrate the Girsanov transformation technique to solve the optimal control problem with partial information in Section 3. Finally, using the theme shown in 3, we solve our original problem considering the exponential-of-integral performance index in Section 4.

Remark 1.1. *For recent advances in numerical methods, readers can go through works such as [6, 7], where, for solving stochastic control problems, particularly Markov chain approximation methods are demonstrated. Some of the applications of stochastic control problems and*

their solutions in mathematical biology, particularly the harvesting and seeding of stochastic populations, are discussed in [5].

2. Full Observation

This section is developed based on the readings from [3, Chapter 3], [9], and [8].

2.1. Problem formulation. Let $W(\cdot)$ be a Wiener process with values in $\mathbb{R}^n$ and covariance matrix $Q(\cdot)$. Consider the following dynamical system,

$$\begin{cases} dX(t) = \Big(F(t)\,X(t) + G(t)\,v(t) + f(t)\Big)\,dt + dW(t), \\ X(0) = X_0. \end{cases} \tag{2.4}$$

Define the filtration

$$\mathcal{F}^t \;=\; \sigma\big\{W(s)\;:\;0 \le s \le t\big\}.$$

An admissible control is a process $v(\cdot)$, adapted to the filtration $\mathcal{F}^t$, such that

$$\int_0^T \|v(t)\|^2\,dt \;<\; \infty \quad \text{(almost surely)}, \quad v(t) \text{ is } \mathcal{F}^t\text{-measurable.} \tag{2.5}$$

We then define our cost functional for the optimal minimization problem as follows,

$$\begin{aligned} J\big(v(\cdot)\big) \;=\; \mathbb{E}\Bigg[\theta\,\exp\theta\Bigg\{ \int_0^T \big(X(t)^* M\,X(t) \;+\; v(t)^* N\,v(t) \;+\; 2\,m\,X(t) \;+\; 2\,n\,v(t)\big)\,dt \\ +\; X(T)^*\,M_T\,X(T) \;+\; 2\,m_T\,X(T)\Bigg\}\Bigg]. \end{aligned} \tag{2.6}$$

where

- M, N, m, n are bounded functions,
- M, M_T are symmetric and nonnegative definite,
- N is symmetric and positive definite,
- $\theta \in \mathbb{R}$.

Remark 2.1. *A few observations and interpretations:*

- $J\big(v(\cdot)\big)$ *can be* $+\infty$.
- *If we set*

$$\begin{aligned} \mathcal{X} := \Bigg\{ \int_0^T \big(X(t)^* M\,X(t) \;+\; v(t)^* N\,v(t) \;+\; 2\,m\,X(t) \;+\; 2\,n\,v(t)\big)\,dt \\ +\; X(T)^*\,M_T\,X(T) \;+\; 2\,m_T\,X(T)\Bigg\}, \end{aligned}$$

  then (2.20) *becomes*

$$J(v(\cdot)) = \mathbb{E}[\theta e^{\theta \mathcal{X}}]. \tag{2.7}$$

- *We can interpret* $\mathcal{X}$ *as a random cost and the cost functional* (2.7) *as the utility of the random cost.*

- *Since*

$$J\big(v(\cdot)\big) \;=\; \theta \;+\; \theta^2\,\mathbb{E}\big[\mathcal{X}\big] \;+\; \theta^3\,\mathbb{E}\big[\mathcal{X}^2\big]\int_0^1\int_0^1 X\,\exp\big(Y\theta\mathcal{X}\big)\,dXdY, \tag{2.8}$$

we can view θ as a measure of risk aversion:

1. *For $\theta > 0$, the second-order moment induces risk aversion.*
2. *For $\theta < 0$, the second-order moment induces risk preference.*

Next we define the stochastic process h_t for any admissible control $v(\cdot)$,

$$\begin{aligned} h_t = &\int_0^t (X^*MX + v^*Nv + 2mX + 2nv)ds \\ &+ X_t^*S_tX_t + 2s_tX_t + \rho_t \end{aligned} \tag{2.9}$$

where S_t, s_t are deterministic functions, such that

- $S_T = M_T, \quad s_T = m_T, \quad \rho_T = 0,$
- $\dot{S}_t, \dot{s}_t, \dot{\rho}_t$ are L^2 functions,
- S_t is symmetric.

In differential form, we have

$$\begin{aligned} dh_t = &\Big[X_t^*(\dot{S}_t + S_tF_t + F_t^*S_t + M_t)X_t \\ &+ v_t^*N_tv_t + 2v_t^*G_t^*S_tX_t + 2(\dot{s}_t + s_tF_t + f_t^*S_t + m_t)X_t \\ &+ 2(n_t + s_tG_t)v_t + \dot{\rho}_t + \mathrm{tr}(S_tQ_t + 2s_tf_t)\Big]dt \\ &+ 2(X_t^*S_t + s_t)dW_t \end{aligned} \tag{2.10}$$

Consequently, by Itô's formula,

$$\begin{aligned} d(e^{\theta h_t}) = \theta^2\exp\theta h_t\Bigg\{ &X_t^*(\dot{S}_t + S_tF_t + F_t^*S_t \\ &- S_t(G_tN_t^{-1}G_t^* - 2\theta Q_t)S_t + M_t)X_t \\ &+ (v_t + N_t^{-1}G_t^*S_tX_t + N_t^{-1}(G_t^*s_t^* + n_t^*))^* \cdot N_t(v_t + N_t^{-1}G_t^*S_tX_t \\ &\quad + N_t^{-1}(G_t^*s_t^* + n_t^*)) \\ &+ 2[\dot{s}_t + s_t(F_t + 2\theta Q_tS_t) + f_t^*S_t + m_t \\ &- (s_tG_t + n_t)N_t^{-1}G_t^*S_t]X_t + \dot{\rho}_t + \mathrm{tr}(S_tQ_t) \\ &+ 2\theta s_tQ_ts_t^* + 2s_tf_t - (s_tG_t + n_t)N_t^{-1}(G_t^*\dot{s}_t + \dot{n}_t)\Bigg\}dt \\ &+ 2\theta^2\exp\theta h_t(X_t^*S_t + s_t)dW_t \end{aligned} \tag{2.11}$$

From the above, we suppose that S_t, s_t, and ρ_t can be calculated using the following equations

$$\dot{S}_t + S_tF_t + F_t^*S_t - S_t(G_tN_t^{-1}G_t^* - 2\theta Q_t)S_t + M_t = 0 \tag{2.12}$$

$$\dot{s}_t + s_t(F_t + 2\theta Q_tS_t - G_tN_t^{-1}G_t^*S_t) + f_t^*S_t + m_t - n_tN_t^{-1}G_t^*S_t = 0 \tag{2.13}$$

$$\dot{\rho}_t + \mathrm{tr}(S_tQ_t) + 2\theta s_tQ_ts_t^* + 2s_tf_t - (s_tG_t + n_t)N_t^{-1}(G_t^*\dot{s}_t + \dot{n}_t) = 0 \tag{2.14}$$

Then (2.11) becomes

$$\begin{aligned} d\big(e^{\theta h_t}\big) &= \theta^2 e^{\theta h_t}\big[v_t + N_t^{-1}G_t^* S_t X_t + N_t^{-1}(G_t^*\dot{s}_t + \dot{n}_t)\big]^* \\ &\quad \times N_t\big[v_t + N_t^{-1}G_t^* S_t X_t + N_t^{-1}(G_t^*\dot{s}_t + \dot{n}_t)\big]dt \\ &\quad + 2\theta^2 e^{\theta h_t}(X_t^* S_t + s_t)dW_t \end{aligned}$$

Thus, we also have

$$\begin{aligned} \theta e^{\theta h_t} &= \theta e^{\theta h_0}\exp\Big\{\theta\int_0^t \big[v_t + N_t^{-1}G_t^* S_t X_t + N_t^{-1}(G_t^*\dot{s}_t + \dot{n}_t)\big]^* \\ &\quad \times N_t\big[v_t + N_t^{-1}G_t^* S_t X_t + N_t^{-1}(G_t^*\dot{s}_t + \dot{n}_t)\big]dt\Big\} \\ &\quad \times \exp\Big\{2\theta\int_0^t (X_t^* S_t + s_t)dW_t - 2\theta^2\int_0^t (X_t^* S_t + s_t)Q_t(S_t X_t + s_t^*)dt\Big\}. \end{aligned} \tag{2.15}$$

2.2. Optimal control. Assume that

- there exists a unique solution of (2.12) with the given terminal conditions on the interval $[0, T]$;
- $S(t)$ is symmetric and nonnegative.

Then we obtain results for the following class of admissible controls

$$v(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^k), \tag{2.16}$$

where $L^2_{\mathcal{F}}(0, T; \mathbb{R}^k)$ denotes the space of square-integrable processes adapted to filtration $\{\mathcal{F}^t\}_{t\in[0,T]}$, and

$$\mathbb{E}\exp\left[2\theta\int_0^T (X_t^* S_t + s_t)\, dW_t \;-\; 2\theta^2\int_0^T (X_t^* S_t + s_t)\, Q_t\, (S_t X_t + s_t^*)\, dt\right] = 1. \tag{2.17}$$

Define now a particular control $u(t)$ by

$$u(t) = -\,N_t^{-1}\big(G_t^* S_t Y_t + G_t^* s_t^* + n_t^*\big), \tag{2.18}$$

with $Y(t)$ defined by

$$\begin{aligned} dY_t &= \Big((F_t - G_t N_t^{-1} G_t^* S_t)\, Y_t \;-\; G_t N_t^{-1}(G_t^* s_t^* + n_t^*) \;+\; f_t\Big)dt \;+\; dW_t, \\ Y(0) &= X_0. \end{aligned} \tag{2.19}$$

We now state and prove our main result.

Theorem 2.1

Let $W(\cdot)$ be a Wiener process with values in $\mathbb{R}^n$ and covariance matrix $Q(\cdot)$. Assume M, N, m, n are bounded functions, M, M_T are symmetric nonnegative definite, N is symmetric positive definite, and $\theta \in \mathbb{R}$. Further suppose that there exists a unique solution of (2.12) under terminal conditions on the interval $[0, T]$, and $S(t)$ is symmetric and nonnegative.
Then the control $u(\cdot)$, defined by (2.18) and (2.19), belongs to the class (2.16) and is optimal in this class for the payoff (2.20) given by

$$J\big(v(\cdot)\big) \;=\; \mathbb{E}\Bigg[\theta\,\exp\theta\Bigg\{\int_0^T \big(X(t)^* M\,X(t) \;+\; v(t)^* N\,v(t) \;+\; 2\,m\,X(t) \;+\; 2\,n\,v(t)\big)\,dt \\ +\; X(T)^*\,M_T\,X(T) \;+\; 2\,m_T\,X(T)\Bigg\}\Bigg]. \tag{2.20}$$

Proof. Let $\hat{h}_t$ denote the process h_t corresponding to the choice $v_t = u_t$. From (2.15), we obtain

$$\theta\exp(\theta\hat{h}_T) = \theta\exp(\theta h_0)\,\exp\Bigg\{2\theta\int_0^T (Y_t^* S_t + s_t)\,dW_t \;-\; 2\theta^2\int_0^T (Y_t^* S_t + s_t)\,Q_t\,(S_t Y_t + s_t^*)\,dt\Bigg\}.$$

If (2.17) holds, then it follows that

$$\mathbb{E}\big[\theta\exp(\theta\hat{h}_T)\big] = \theta\exp(\theta h_0),$$

i.e.,

$$J(u(\cdot)) = \theta\exp(\theta h_0). \tag{2.21}$$

Moreover, for any control $v(\cdot)$ in the class (2.16), we have from (2.15)

$$\theta\exp(\theta h_T) \geq \theta\exp(\theta h_0)\,\exp\Bigg\{2\theta\int_0^T (X_t^* S_t + s_t)\,dW_t \;-\; 2\theta^2\int_0^T (X_t^* S_t + s_t)\,Q_t\,(S_t X_t + s_t^*)\,dt\Bigg\}.$$

Hence, by (2.16),

$$\mathbb{E}\big[\theta\exp(\theta h_T)\big] \geq \theta\exp(\theta h_0),$$

i.e.,

$$J(v(\cdot)) \geq \theta\exp(\theta h_0).$$

Comparing with (2.21), optimality follows, provided (2.17) holds. We now verify (2.17).

Claim.

$$\mathbb{E}\exp\Bigg[2\theta\int_0^T (Y_t^* S_t + s_t)\,dW_t \;-\; 2\theta^2\int_0^T (Y_t^* S_t + s_t)\,Q_t\,(S_t Y_t + s_t^*)\,dt\Bigg] = 1.$$

Set

$$\varphi_t := 2\theta\big(S_t Y_t + s_t^*\big).$$

From (2.19), φ_t is a Gaussian process (with bounded in t mean and covariance m_t and Π_t, respectively). Let $\delta > 0$ be such that

$$I - 2\delta\Pi_t \;\geq\; \beta I, \quad \beta > 0, \quad \forall\, t \in [0, T].$$

Then a direct calculation yields

$$\mathbb{E}\exp\big(\delta\,|\varphi_t|^2\big) = \frac{\exp\Big[\delta|m_t|^2 + 2\delta^2\, m_t^*\,\Pi_t^{1/2}\,(I-2\delta\Pi_t)^{-1}\,\Pi_t^{1/2} m_t\Big]}{\sqrt{|I-2\delta\Pi_t|}} \;\le\; C,$$

for some constant C independent of t. This exponential integrability criterion is sufficient to ensure (2.17) (see [4]). □

We now demonstrate the Girsanov transformation technique, closely following [3, Chapter 2], which will be used later to solve the problem under partial observation.

3. Optimal stochastic control problem with partial information

3.1. Model. Let us assume there exists

- ξ, a random variable with Gaussian probability law with mean X_0 and covariance matrix P_0,
- $W(\cdot)$, a Wiener process with covariance $Q(\cdot)$;

$$\xi \text{ is independent of } W(\cdot). \tag{3.22}$$

- $Z(\cdot)$, a Wiener process with covariance $R(\cdot)$, independent of ξ, $W(\cdot)$, where $R(t) \ge \rho I$, $\rho > 0$, $\forall t \in [0,T]$.

Next, we define filtration,

$$\mathcal{F}^t = \sigma(\xi, W(s), Z(s), s \le t), \quad \mathcal{Z}^t = \sigma(Z(s), s \le t).$$

and admissible controls as being square integrable processes $v(\cdot) \in \mathcal{U} := L^2_{\mathcal{Z}}(0,T;\mathbb{R}^k)$, that is, adapted to this family $\mathcal{Z}^t$.

For any admissible control $v(\cdot)$, define the state $X(\cdot)$ by

$$\begin{aligned} dX &= (F(t)X + G(t)v + f(t))dt + dW \\ X(0) &= \xi \end{aligned} \tag{3.23}$$

The $\mathcal{F}^t-$adapted process $X(\cdot)$ depends on $v(\cdot)$.

Let us next define

$$\eta^v(t) = \exp\left[\int_0^t (X^*H^* + h^*)R^{-1}dz - \frac{1}{2}\int_0^t (X^*H^* + h^*)R^{-1}(HX+h)ds\right] \tag{3.24}$$

$$b^v(t) = Z(t) - \int_0^t (HX(s) + h(s))ds. \tag{3.25}$$

The index $v = v(\cdot)$ means that the processes η, b depend on $v(\cdot)$ (via of course the state $X(\cdot)$).

The process η^v satisfies

$$d\eta^v = \eta^v(X^*H^* + h^*)R^{-1}dz, \quad \eta^v(0) = 1. \tag{3.26}$$

We can say (cf. [4]) that

$$\mathbb{E}\eta^v(t) \le 1, \tag{3.27}$$

Therefore $\eta^v(T)$ does not represent the Radon–Nikodym derivative of a probability P^v such that

$$\frac{dP^v}{dP} = \eta^v(T). \tag{3.28}$$

Although, irrelevent for the discussed control problem, for integrability conditions to be satisfied, we must restrict

$$\mathbb{E}\eta^v(T) = 1. \tag{3.29}$$

We begin by writting,

$$X(t) = \alpha(t) + x_1(t) \tag{3.30}$$

Define the cost functional as,

$$\begin{aligned} J(v(\cdot)) = \mathbb{E}\int_0^T \eta^v(t)(X^*MX + v^*Nv + 2mX + 2nv)dt \\ + \mathbb{E}\eta^v(T)(X(T)^*M_TX(T) + 2m_TX(T)) \end{aligned} \tag{3.31}$$

Note taht cost functional can acheive value $+\infty$.

Claim: (3.64) is well-defined. This can be seen from the fact that bound

$$|x_1(t)| \leq C\left(\int_0^T |v|^2 ds\right)^{1/2} \tag{3.32}$$

along with invertibility of N, implies admissibility of

$$\mathbb{E}\int_0^T \eta^v(t)(X^*MX + v^*Nv + 2mX + 2nv)dt + \mathbb{E}\eta^v(T)(X(T)^*M_TX(T) + 2m_TX(T))$$

Lemma

$$\mathbb{E}\eta^v(t)|\alpha(t)|^4 \leq C \tag{3.33}$$

with a constant independent of the control.

Proof. Lets, consider approximation $\eta_v^\epsilon(t)$ of η^v, as follows

$$\eta_v^\epsilon(t) = \exp\left[\int_0^t (x_\epsilon^*H^* + h^*)R^{-1}dz - \frac{1}{2}\int_0^t (x_\epsilon^*H^* + h^*)R^{-1}(Hx_\epsilon + h)ds\right]$$

where

$$x_\epsilon(t) = \frac{X(t)}{(1+\epsilon|X(t)|^2)^{1/2}}.$$

We can extract a subsequence, (denoted by same ϵ, for sake of saving notation), s. t.

$$\text{almost surely} \quad \eta_v^\epsilon(t) \to \eta^v(t), \quad \forall t.$$

Now

$$d|\alpha(t)|^4 = 4|\alpha(t)|^2\alpha^*[(F\alpha + f)dt + dW] + 2|\alpha(t)|^2\,\mathrm{tr}(Q + 2\alpha^*Q\alpha)dt$$

and

$$\begin{aligned} d\eta_v^\epsilon(t)|\alpha(t)|^4 = 4\eta_v^\epsilon(t)|\alpha(t)|^2\alpha^*[(F\alpha + f)dt + dW] + 2\eta_v^\epsilon(t)|\alpha(t)|^2\,\mathrm{tr}(Q + 2\alpha^*Q\alpha)dt \\ + \eta_v^\epsilon(t)|\alpha(t)|^4(x_\epsilon^*H^* + h^*)R^{-1}dz. \end{aligned}$$

From above we immediately obtain,

$$\mathbb{E}\eta_v^\epsilon(t)|\alpha(t)|^4 = \mathbb{E}|\xi|^4 + 4\mathbb{E}\int_0^t \eta_v^\epsilon(s)|\alpha(s)|^2\alpha^*(s)(F\alpha + f)ds$$

$$+ 2\mathbb{E}\int_0^t \eta_v^\epsilon(s)|\alpha(s)|^2 \operatorname{tr}(Q + 2\alpha^* Q\alpha)ds.$$

Since $\mathbb{E}\eta_v^\epsilon(s) = 1\ \forall s$, above implies

$$\mathbb{E}\eta_v^\epsilon(t)|\alpha(t)|^4 \leq \mathbb{E}|\xi|^4 + C\left(1 + \mathbb{E}\int_0^t \eta_v^\epsilon(s)|\alpha(s)|^4 ds\right)$$

where C is a constant independent of ϵ and $v(\cdot)$.

Gronwall's lemma immediately ensures bound on approximation, that is,

$$\mathbb{E}\eta_v^\epsilon(t)|\alpha(t)|^4 \leq C \tag{3.34}$$

Taking $\epsilon \longrightarrow 0$ and passing limits via virtue of Fatou's lemma, required bound (3.33) is established. □

Goal: Minimise $J(v(\cdot))$ defined by (3.64) among all $v(\cdot) \in \mathcal{U}$.

3.2. Primary results. Observe that $\alpha(\cdot)$ is the solution of

$$\begin{aligned} d\alpha &= (F\alpha + f)dt + dW, \\ \alpha(0) &= \xi. \end{aligned} \tag{3.35}$$

Denoting

$$\begin{aligned} \frac{dz_1}{dt} &= H(t)x_1(t), \\ z_1(0) &= 0 \end{aligned} \tag{3.36}$$

we obtain,

$$Z(t) = z_1(t) + \beta^v(t) \tag{3.37}$$

$$\begin{aligned} d\beta^v &= (H(t)\alpha(t) + h)dt + db^v, \\ \beta^v(0) &= 0 \end{aligned} \tag{3.38}$$

Defining filtration as

$$\mathcal{F}_v^t = \sigma(\beta^v(s), s \leq t); \tag{3.39}$$

we have from (3.37) that

$$\mathcal{F}_v^t \subset \mathcal{Z}^t. \tag{3.40}$$

Lemma

If $v(\cdot) \in L^2_{\eta^v}(0, T; \mathbb{R}^k)$, i.e. $\mathbb{E}\int_0^T \eta^v(t)|v(t)|^2 dt < \infty$, then $\eta^v(t)$ is a $\mathcal{F}^t$ martingale.

Proof. Let $v(\cdot)$)

$$\eta_\epsilon(t) = \frac{\eta(t)}{1 + \epsilon\eta(t)};$$

then

$$d\eta_\epsilon = (1 + \epsilon\eta)^{-2}\eta(X^*H^* + h^*)R^{-1}dz - \epsilon(1 + \epsilon\eta)^{-3}\eta(X^*H^* + h^*)R^{-1}(HX + h)dt$$

which implies

$$\mathbb{E}\eta_\epsilon(t) = 1 - \epsilon\mathbb{E}\int_0^t (1 + \epsilon\eta)^{-3}\eta^2(X^*H^* + h^*)R^{-1}(HX + h)ds. \tag{3.41}$$

Clearly

$$H_\epsilon = \epsilon(1+\epsilon\eta)^{-3}\eta^2(X^*H^*+h^*)R^{-1}(HX+h) \quad \text{a.e. a.s.}$$

$$0 \le H_\epsilon \le \eta(X^*H^*+h^*)R^{-1}(HX+h).$$

And we also have,

$$\begin{aligned}\mathbb{E}\int_0^T \eta(X^*H^*+h^*)R^{-1}(HX+h)dt &\le C\mathbb{E}\int_0^T \eta(t)\left(\int_0^t |v(s)|^2 ds\right)^{1/2} dt \\ &\quad + C\mathbb{E}\int_0^T \eta(t)|\alpha(t)|^2 dt \\ &\le C\left[\mathbb{E}\int_0^T \eta(t)\left(\int_0^t |v(s)|^2 ds\right)ds\right]^{1/2} + C \\ &\le C\left[\mathbb{E}\int_0^T \eta(t)|v(t)|^2 dt\right]^{1/2} + C \\ &\le C\end{aligned}$$

for controls in $L^2_{\eta^v}(0,T;\mathbb{R}^k)$.

By the virtue of Lebesgue's dominated convergengce theorem we can pass the limit to imply below, which upon application of fact that $\eta(t)$ is an $\mathcal{F}^t$ upper martingale implies that it is an $\mathcal{F}^t$ martingale, and we are done

$$\mathbb{E}\int_0^t H_\epsilon ds \to 0 \quad \text{and} \quad \mathbb{E}\eta(t) = 1.$$

□

Now, consider

$$\hat{x}(t) = \mathbb{E}^v[X(t) \mid \mathcal{Z}^t] \tag{3.42}$$

then from (3.30) one has

$$\hat{x}(t) = x_1(t) + \mathbb{E}^v[\alpha(t) \mid \mathcal{Z}^t]. \tag{3.43}$$

We shall prove the important lemma,

Lemma

The following property holds:

$$\mathbb{E}^v[\alpha(t) \mid \mathcal{Z}^t] = \mathbb{E}^v[\alpha(t) \mid \mathcal{F}^t_v]. \tag{3.44}$$

Proof. We first prove above result for $t = T$. From (3.28) (definition of P^v) we have,

$$\mathbb{E}^v[\alpha(T) \mid \mathcal{Z}^T] = \frac{\mathbb{E}[\eta^v(T)\alpha(T) \mid \mathcal{Z}^T]}{\mathbb{E}[\eta^v(T) \mid \mathcal{Z}^T]}. \tag{3.45}$$

We then can express,

$$\begin{aligned}\eta^v(T) = \exp\Bigg[&\int_0^T x_1^* H^* R^{-1} dz - \frac{1}{2}\int_0^T x_1^* H^* R^{-1} H x_1 dt \\ &+ \int_0^T (\alpha^* H^* + h^*)R^{-1} dz - \frac{1}{2}\int_0^T (\alpha^* H^* + h^*)R^{-1}(H\alpha + h)dt\end{aligned}$$

$$- \int_0^T x_1^* H^* R^{-1}(H\alpha + h)dt\Big].$$

As integrals are $\mathcal{Z}^T$-measurable, from (3.45), we get

$$\mathbb{E}^v[\alpha(T) \mid \mathcal{Z}^T] = \frac{\mathbb{E}[\alpha(T)\Lambda^v(T) \mid \mathcal{Z}^T]}{\mathbb{E}[\Lambda^v(T) \mid \mathcal{Z}^T]} \tag{3.46}$$

where

$$\begin{aligned}\Lambda^v(T) &:= \exp\Big[\int_0^T (\alpha^* H^* + h^*)R^{-1}dz - \frac{1}{2}\int_0^T (\alpha^* H^* + h^*)R^{-1}(H\alpha + h)dt \\ &\quad - \int_0^T x_1^* H^* R^{-1}(H\alpha + h)dt\Big] \\ &= \exp\Big[\int_0^T (\alpha^* H^* + h^*)R^{-1}d\beta_v - \frac{1}{2}\int_0^T (\alpha^* H^* + h^*)R^{-1}(H\alpha + h)dt\Big]. \end{aligned} \tag{3.47}$$

Observe that:

- α is independent of z,
- β_v is $\mathcal{Z}^T$-measurable.

Assuming differentiability of $R^{-1}H$, $R^{-1}h$, we have,

$$\begin{aligned}\Lambda^v(T) = \exp\Bigg[&(\alpha^*(T)H^*(T) + h^*(T))R^{-1}(T)\beta_v(T) \\ &- \int_0^T \beta_v^*(t)(R^{-1}Hd\alpha + (R^{-1}H)'\alpha dt + (R^{-1}h)'dt)\Bigg]\end{aligned}$$

which implies

$$\mathbb{E}[\Lambda^v(T) \mid \mathcal{Z}^T] \text{ is } \mathcal{F}_v^T \text{ measurable.}$$

hence, from (3.45)

$$\mathbb{E}^v[\alpha(T) \mid \mathcal{Z}^T] \text{ is } \mathcal{F}_v^T \text{ measurable.} \tag{3.48}$$

Due to inclusion results of (3.48) and (3.40)

$$\mathbb{E}^v\big[\mathbb{E}^v[\alpha(T) \mid \mathcal{Z}^T] \mid \mathcal{F}_v^T\big] = \mathbb{E}^v[\alpha(T) \mid \mathcal{Z}^T] = \mathbb{E}^v[\alpha(T) \mid \mathcal{F}_v^T] = \hat{\alpha}(T).$$

□

3.3. Optimal control. Refer to Kalman filter theory [3, Chapter 1], which implies that $\hat{\alpha}$ is the solution of

$$d\hat{\alpha} = (F\hat{\alpha} + f)dt + PH^*R^{-1}(d\beta_v - (H\hat{\alpha} + h)dt), \quad \hat{\alpha}(0) = x_0. \tag{3.49}$$

From above(3.49) and (3.43), we see that $\hat{x}(t)$ is the solution of

$$\begin{aligned}d\hat{x} &= \big[(F - PH^*R^{-1}H)\hat{x} + f - PH^*R^{-1}h + Gv\big]dt + PH^*R^{-1}dz, \\ \hat{x}(0) &= x_0. \end{aligned} \tag{3.50}$$

where the noted error $\epsilon = x - \hat{x}$ satisfies

$$d\epsilon = (F - PH^*R^{-1}H)\epsilon dt + dw - PH^*R^{-1}db_v, \quad \epsilon(0) = \xi - x_0. \tag{3.51}$$

Hence, ϵ is a Gaussian process $\mathcal{N}(0, P(\cdot))$.

It follows that cost functional

$$J(v(\cdot)) = \mathbb{E}^v\Big[\int_0^T (\hat{x}^* M\hat{x} + v^* N v + 2m\hat{x} + 2nv)dt + \hat{x}(T)^* M_T \hat{x}(T) + 2m_T\hat{x}(T)\Big] + \int_0^T \mathrm{tr}(MP)dt + \mathrm{tr}(M_T P(T)). \tag{3.52}$$

Next, cosidering the innovation process as,

$$\begin{aligned} I_v(t) &= z(t) - \int_0^t (H(s)\hat{x}(s) + h)ds \\ &= b_v(t) - \int_0^t (H(s)\hat{\alpha}(s) + h)ds. \end{aligned}$$

Note that,

$$\mathbb{E}^v[(I_v(t) - I_v(s)) \mid \mathcal{Z}^s] = 0 \quad \forall t \geq s$$

$$\mathbb{E}^v[(I_v(t) - I_v(s))(I_v(t) - I_v(s))^* \mid \mathcal{Z}^s] = \int_s^t R(\lambda)d\lambda.$$

Hence, from the filtering theory (see lecture otes 21-22), we know that $I_v(t)$ is (for P^v) a Wiener process with covariance $R(\cdot)$. It is $\mathcal{F}_v^t-$ Brownian, and a $\mathcal{Z}^t-$ Wiener process as well.

3.3.1. *Problem reformulation.* Problem can be formulated as full observation problem as following

$$d\hat{x} = (F\hat{x} + f + Gv)dt + PH^*R^{-1}dI_v, \quad \hat{x}(0) = x_0. \tag{3.53}$$

with the payoff

$$J_1(v(\cdot)) = \mathbb{E}^v\Big[\int_0^T \Big(\hat{X}^* M\hat{X} + v^* N v + 2m\hat{X} + 2nv\Big)dt + \hat{X}(T)^* M_T\hat{X}(T) + 2m_T\hat{X}(T)\Big]. \tag{3.54}$$

This problem is a stochastic control problem with full observation; where $\mathcal{Z}^t-$ adapted $z(\cdot)$ is the unique noise.

The problem still persist as P^v varies with $v(\cdot)$, which can be delt by setting

$$v(t) = -N^{-1}(t)\Big[n^*(t) + G^*(t)\Big(\Pi(t)\hat{X}(t) + r(t)\Big)\Big] + \mu(t) \tag{3.55}$$

and then consider

- $\eta(\cdot)$ be the solution of

$$d\eta = \big[(F - GN^{-1}G^*\Pi)\eta + f - GN^{-1}(n^* + G^*r)\big]dt + PH^*R^{-1}dI_v, \quad \eta(0) = x_0 \tag{3.56}$$

- $$\hat{X} - \eta = \tilde{X}, \quad \chi = -N^{-1}[n^* + G^*(\Pi\eta + r)]$$

$$v = \chi - N^{-1}G^*\Pi\tilde{X} + \mu. \tag{3.57}$$

A simple computation yeilds

$$J_1(v(\cdot)) = \mathbb{E}^v\Big[\int_0^T (\eta^* M\eta + \chi^* N\chi + 2m\eta + 2n\chi)dt + \eta(T)^* M_T\eta(T) + 2m_T\eta(T)$$

$$+\int_0^T \big\{\hat{X}^*M\tilde{X}+(-\tilde{X}^*\Pi GN^{-1}+\mu^*)N(-N^{-1}G^*\Pi\tilde{X}+\mu)$$
$$+[-\tilde{X}(T)^*\Pi(T)G(T)N^{-1}(T)+\mu^*(T)]M_T[-N^{-1}(T)G^*(T)\Pi(T)\tilde{X}(T)+\mu(T)]\big\}dt\Big]$$

which implies

$$J_1(v(\cdot))\geq \mathbb{E}^v\Big[\int_0^T(\eta^*M\eta+\chi^*N\chi+2m\eta+2n\chi)dt+\eta(T)^*M_T\eta(T)+2m_T\eta(T)\Big]. \quad (3.58)$$

And we have

$$d[\eta(t)^*\Pi(t)\eta(t)+2r(t)^*\eta(t)]=\Big[-(\eta^*M\eta+2m\eta+\chi^*N\chi+2n\chi)+2f^*r-(r^*G+n)N^{-1}(G^*r+n^*)$$
$$+\text{tr }R^{-1}HP\Pi H^*\Big]dt+2(\eta^*\Pi+r^*)PH^*R^{-1}dI_v$$

which upon application of bound (3.58) implies

$$J_1(v(\cdot))\geq\int_0^T\big[2f^*r-(r^*G+n)N^{-1}(G^*r+n^*)+\text{tr}(R^{-1}HP\Pi H^*)\big]dt. \quad (3.59)$$

Consider $\hat{y}(t)$ adapted to $\mathcal{Z}^t$ be the process satisfying,

$$\begin{cases} d\hat{y}(t)=(F-PH^*R^{-1}H)\hat{y}(t)+f(t)-PH^*R^{-1}h(t)\\ \qquad\quad -GN^{-1}n^*(t)-GN^{-1}G^*(\Pi(t)\hat{y}(t)+r(t))\\ \qquad\quad +PH^*R^{-1}dZ(t),\\ \hat{y}(0)=x_0. \end{cases} \quad (3.60)$$

We now define control $u(\cdot)\in\mathcal{U}$ by

$$u(t)=-N^{-1}(t)[n^*(t)+G^*(t)(\Pi(t)\hat{y}(t)+r(t))]. \quad (3.61)$$

Define $y(\cdot)$ to be the state corresponding to the control $u(\cdot)$, and $\eta^u(\cdot)$ associated to $u(\cdot)$ by (cf. (3.26))

$$\begin{cases} d\eta^u(t)=\eta^u(t)(y(t)^*H(t)^*+h(t)^*)R(t)^{-1}dz(t),\\ \eta^u(0)=1. \end{cases} \quad (3.62)$$

Lemma

Following estimate holds:

$$\mathbb{E}\eta^u(t)|\hat{y}(t)|^2\leq C. \quad (3.63)$$

Proof. Set $y_\epsilon(t)$ by

$$y_\epsilon(t):=\frac{y(t)}{(1+\epsilon|y(t)|^2)^{1/2}},$$

Replacing y with y_ϵ, we define η^u_ϵ as in accord with (3.62) and note that

$$d|\hat{y}(t)|^2=2\hat{y}^*\big\{(F-PH^*R^{-1}H)\hat{y}-PH^*R^{-1}h+f-GN^{-1}n^*-GN^{-1}G^*(\Pi\hat{y}+r)\big\}dt$$
$$+PH^*R^{-1}HPdt,$$

$$d\eta^u_\epsilon(t)|\hat{y}(t)|^2=\eta^u_\epsilon|\hat{y}(t)|^2(y^*_\epsilon H^*+h^*)R^{-1}dz+2\eta^u_\epsilon y^*_\epsilon PH^*R^{-1}dz$$

$$+\eta_\epsilon^u\Big[2y_\epsilon^*\big\{(F-PH^*R^{-1}H)\hat{y}-PH^*R^{-1}h+f-GN^{-1}n^*-GN^{-1}G^*(\Pi\hat{y}+r)\big\}$$
$$+PH^*R^{-1}HP+2(y_\epsilon^*H^*+h^*)R^{-1}HP\hat{y}\Big]dt.$$

Further, we have,

- $\mathbb{E}|\hat{y}(t)|^4<\infty$
- $\mathbb{E}(\eta_\epsilon^u)^2<\infty$

Above implies,

$$\mathbb{E}\eta_\epsilon^u(t)|\hat{y}(t)|^2=|x_0|^2+\mathbb{E}\int_0^t\eta_\epsilon^u\Big\{2\hat{y}^*[(F-PH^*R^{-1}H)\hat{y}-PH^*R^{-1}h+f$$
$$-GN^{-1}n^*-GN^{-1}G^*(\Pi\hat{y}+r)]+PH^*R^{-1}HP+2(y_\epsilon^*H^*+h^*)R^{-1}HP\hat{y}\Big\}ds$$
$$\leq C+\mathbb{E}\int_0^t\eta_\epsilon^u(s)|\hat{y}(s)|^2ds.$$

Again as in previous lemmas, upon application of Gronwall's inequality and Fatou's lemma we get our desired result. □

From Lemma 3.3.1 and defined control (3.61) we can see that $u(\cdot)\in\mathcal{U}$. Therefore, $\eta^u(t)$ is a $\mathcal{F}^t$ martingale and P^u is a probability.

We can the get $\hat{y}(t)$, by taking into account of $v(\cdot)=u(\cdot)$, and (3.60), we obtain that

$$\hat{y}(t)=\mathbb{E}^u[y(t)|\mathcal{Z}^t].$$

Considering the innovation process,

$$I_u(t)=z(t)-\int_0^t(H\hat{y}+h)ds$$

we find that $\eta(\cdot)$, the solution of (3.56) with $v(\cdot)=u(\cdot)$, coincides with $\hat{y}$.

Thus, all of above implies:

- $\tilde{x}=\hat{y}-\eta=0$
- $\chi=u,\quad \mu=0.$
-

$$J_1(u(\cdot))=\mathbb{E}^u\left[\int_0^T(\hat{y}^*M\hat{y}+u^*Nu+2m\hat{y}+2nu)dt+y(T)^*M_Ty(T)+2m_Ty(T)\right]$$
$$=\int_0^T\big[2f^*r-(r^*G+n)N^{-1}(G^*r+n^*)+\mathrm{tr}(R^{-1}HP\Pi H^*)\big]dt.$$

From (3.59), we finally get

$$J_1(v(\cdot))\geq J_1(u(\cdot)),\quad \forall v(\cdot).$$

Now we will state our main result as,

Theorem 3.1

The control $u(\cdot)$ defined by (3.61) is optimal for the problem of minimizing (3.64), that is,

$$J(v(\cdot)) = \mathbb{E}\int_0^T \eta^v(t)(X^*MX + v^*Nv + 2mX + 2nv)dt + \mathbb{E}\eta^v(T)(X(T)^*M_TX(T) + 2m_TX(T)) \tag{3.64}$$

on the space $\mathcal{U}$.

4. Partial Observation

This section is modeled on [3], in which we will deal with a problem similar to that described in Section 2 for the full observation case. We will employ the technique we described in Section 3, based on the Girsanov transformation, as below.

4.1. Problem settings. Let $(\Omega, \mathcal{A}, P)$ be a convenient probability space, in which we define the initial setup as we did in the previous section.

- ξ, a random variable with Gaussian probability law, with mean x_0 and covariance matrix P_0
- $W(\cdot)$, a Wiener process with covariance matrix $Q(\cdot)$
- $z(\cdot)$, a Wiener process with covariance matrix $R(\cdot)$

The processes $W(\cdot)$, $z(\cdot)$, and the variable ξ are mutually independent.

$$R(t) \geq \rho I, \quad \rho > 0, \quad \forall t \in [0, T] \tag{4.65}$$

We define

$$\mathcal{F}^t = \sigma(\xi, W(s), z(s), s \leq t), \quad \mathcal{Z}^t = \sigma(z(s), s \leq t).$$

An admissible control is for the time being a process belonging to the space

$$\mathcal{U} = L^2_{\mathcal{Z}}(0, T; \mathbb{R}^k). \tag{4.66}$$

Next, define followings:

- the state $X(\cdot)$ is adapted to $\mathcal{F}^t$ for admissible control $v(\cdot)$, by

$$dX = (F(t)X + G(t)v + f(t))dt + dW,$$
$$X(0) = \xi. \tag{4.67}$$

- processes $\eta^v(t)$ and $b^v(t)$(depending on $v(\cdot)$ as is the case for $X(\cdot)$),

$$\eta^v(t) = \exp\left[\int_0^t (X^*H^* + H^*)R^{-1}dz - \frac{1}{2}\int_0^t (X^*H^* + H^*)R^{-1}(HX + H)ds\right] \tag{4.68}$$

$$b^v(t) = z(t) - \int_0^t (HX(s) + H(s))ds. \tag{4.69}$$

Further note, that

- process η^v satisfies

$$d\eta^v = \eta^v(X^*H^* + H^*)R^{-1}dz$$
$$\eta^v(0) = 1 \tag{4.70}$$

•

$$\mathbb{E}\eta^v(t) \leq 1 \tag{4.71}$$

Define the positive finite measure P^v by the Radon-Nikodym derivative.

$$\frac{dP^v}{dP} = \eta^v(T) \tag{4.72}$$

Along the simmilar lines of (3.30), we write

$$X(t) = X_1(t) + \alpha(t) \tag{4.73}$$

where

$$\frac{dX_1}{dt} = FX_1 + Gv, \quad X_1(0) = 0 \tag{4.74}$$

$$d\alpha = (F\alpha + f)dt + dW, \quad \alpha(0) = \xi \tag{4.75}$$

The cost functional (well-known exponential payoff) is defined by

$$J(v(\cdot)) = \theta\mathbb{E}\eta^v(T)\exp\left\{\theta\left[\int_0^T (X^*MX + v^*Nv + 2mX + 2nv)dt\right.\right. \tag{4.76}$$

$$\left.\left. + X(T)^*M_TX(T) + 2m_TX(T)\right]\right\} \tag{4.77}$$

All coefficients $F, G, f, M, N, m, n, M_T, m_T$ are the same as in the full observation case. Expectation of positive random variables, which exists and is possibly equal to $+\infty$.

Claim: For the case of $\theta < 0$, i,e; value $-\infty$ is not possible.

Note,

$$\mathbb{E}\eta^v(T)\exp\left\{\theta\left[\int_0^T (X^*MX + v^*Nv + 2mX + 2nv)dt + X(T)^*M_TX(T) + 2m_TX(T)\right]\right\}$$

$$\leq \mathbb{E}\eta^v(T)\exp\left\{\theta\left[\int_0^T (v^*Nv + 2mX_1 + 2nv)dt + 2m_TX_1(T)\right]\right\} \times \exp\left\{\theta\left[\int_0^T 2md\alpha + 2m_T\alpha(T)\right]\right\} \tag{4.78}$$

(4.74) and the invertibility of the matrix N implies that

$$\int_0^T (v^*Nv + 2mX_1 + 2nv)dt + 2m_TX_1(T) \geq -C.$$

Give LHS of (4.78) finite, it suffices to show that

$$\mathbb{E}\eta^v(T)\exp\left\{\theta\left[\int_0^T 2m\alpha dt + 2m_T\alpha(T)\right]\right\} < C_1.$$

A simple application of Cauchy–Schwarz inequality and Jensen's inequality yields following estimates,

$$\mathbb{E}\eta^v\exp\left\{\theta\left[\int_0^T 2m\alpha dt + 2m_T\alpha(T)\right]\right\} \leq \left[\mathbb{E}\eta^v(T)\exp\left(\theta\int_0^T 4m\alpha dt\right)\right]^{1/2}$$

$$\times \left[\mathbb{E}\eta^v(T)\exp(4\theta m_T\alpha(T))\right]^{1/2}$$

$$\mathbb{E}\eta^v(T)\exp\left(\theta\int_0^T 4m\alpha dt\right) \le \frac{1}{T}\mathbb{E}\eta^v(T)\int_0^T \exp(4\theta T m\alpha)dt$$
$$\le \frac{1}{T}\int_0^T \mathbb{E}\eta^v(t)\exp(4\theta T m\alpha)dt$$

As $\eta^v(t)$ is an upper $\mathcal{F}^t$ martingale, we can immediatly see the result provided following lemma holds.

Lemma

Let $k(t) \in (\mathbb{R}^n)^*$ be such that $|k(t)| \le k_0$, then

$$\mathbb{E}\eta^v(t)\exp(k(t)\alpha(t)) \le C, \quad \forall t \in [0,T] \tag{4.79}$$

where C does not depend on $v(\cdot)$.

Proof. In order to save notation take $t = T$. Hence, we wish to show that,

$$\mathbb{E}\eta^v(T)\exp(k\alpha(T)) \le C. \tag{4.80}$$

Define $\nu(t)$ s.t.

$$\dot{\nu} + \nu F = 0, \quad \nu(T) = k,$$

and consider $\eta^v(T)\exp[\nu(t)\alpha(t)]$.

Then, By application of Itô's formula, we have

$$d(\eta^v(t)\exp[\nu(t)\alpha(t)]) = \eta^v(t)\exp[\nu(t)\alpha(t)]\Big[(X^*H^* + H^*)R^{-1}dz$$
$$+ vdW + (\nu f + \tfrac{1}{2}\nu Q\nu^*)dt\Big].$$

Thus,

$$\frac{d(\eta^v(t)\exp[\nu(t)\alpha(t)])}{1+\epsilon\eta^v(t)\exp[\nu(t)\alpha(t)]} = \Big[(1+\epsilon\eta^v\exp\nu\alpha)^{-2}\eta^v\exp\nu\alpha\big[(X^*H^*+H^*)R^{-1}dz$$
$$+ vdW + (\nu f + \tfrac{1}{2}\nu Q\nu^*)dt\big]$$
$$- \epsilon(1+\epsilon\eta^v\exp\nu\alpha)^{-3}\big[(X^*H^*+H^*)R^{-1}(HX+H)$$
$$+ \nu Q\nu^*\big]\eta^v\exp 2\nu\alpha dt\Big],$$

implying,

$$\mathbb{E}\left[\frac{\eta^v(t)\exp[\nu(t)\alpha(t)]}{1+\eta^v(t)\exp[\nu(t)\alpha(t)]}\right] \le \mathbb{E}\left[\frac{\exp[\nu(0)\alpha(0)]}{1+\exp[\nu(0)\alpha(0)]}\right]$$
$$+ \mathbb{E}\int_0^t \frac{(\nu f + \frac{1}{2}\nu Q\nu^*)\eta^v(t)\exp[\nu(t)\alpha(t)]}{\{1+\eta^v(t)\exp[\nu(t)\alpha(t)]\}^2}dt.$$

From above, we obtain

$$\mathbb{E}\left[\frac{\eta^v(t)\exp[\nu(t)\alpha(t)]}{1+\eta^v(t)\exp[\nu(t)\alpha(t)]}\right] \leq C.$$

Letting $\varepsilon \to 0$, the desired result (4.79) follows. □

4.2. Some useful results. For $v(\cdot) \in \mathcal{U}$ (see (4.66)), the random variable is

$$X_v = \theta\eta^v(T)\exp\left\{\theta\left[\int_0^T (X^*MX + v^*Nv + 2mX + 2nv)dt \right. \tag{4.81}$$

$$\left. + X(T)^*M_TX(T) + 2m_TX(T)\right]\right\}. \tag{4.82}$$

From (4.77), we see that $J(v(\cdot)) = \mathbb{E}X_v$ (can be $+\infty$)
Assumptions:

- $$(R^{-1}H)^*, \quad (R^{-1}H)' \in L^\infty(0,T) \tag{4.83}$$
- The following Riccati equation gives continuous solution on $[0,T]$.

$$\dot{\Sigma} + \Sigma F + F^*\Sigma - \Sigma Q\Sigma + H^*R^{-1}H - 2\theta M = 0, \quad \Sigma(T) = -2\theta M_T \tag{4.84}$$

Define:

- $\sigma(t) \in (\mathbb{R}^n)^*$ as,

$$\dot{\sigma} + \sigma(F - Q\Sigma) - z^*[R^{-1}H(F - Q\Sigma) + (R^{-1}H)'] + 2\theta m - (f^* + v^*G^*)\Sigma - H^*R^{-1}H = 0$$
$$\sigma(T) = 2\theta m_T + z^*(T)R^{-1}(T)H(T) \tag{4.85}$$

- $\beta(t) \in \mathbb{R}$ as,

$$\beta(t) = -\int_t^T \Big[2\theta(v^*Nv + 2nv) + z^*R^{-1}HQH^*R^{-1}z - H^*R^{-1}H - 2(H^*R^{-1})'z$$
$$+ Q\sigma^* - 2z^*R^{-1}H(Q\sigma^* + f + Gv) + 2\sigma(f + Gv) - \operatorname{tr}\Sigma Q\Big]ds$$
$$- 2H^*(T)R^{-1}(T)z(T) \tag{4.86}$$

Also, note:

- $\sigma(\cdot)$, $\beta(\cdot)$ depend parametrically on $z(\cdot)$(through $v(\cdot)$),
- $\sigma(\cdot)$, $\beta(\cdot)$ are independent from $\xi, W(\cdot)$.

We can then state our first result,

Lemma

We assume (4.83), (4.84). Then we can express X_v as

$$X_v = \theta\exp\left\{-\frac{1}{2}[\xi^*\Sigma(0)\xi - 2\sigma(0)\xi + \beta(0)]\right\}$$
$$\times \exp\left[\int_0^T -(Z^*R^{-1}H + X^*\Sigma - \sigma)dW - \frac{1}{2}|H^*R^{-1}Z + \Sigma X - \sigma|^2dt\right] \tag{4.87}$$

Proof. For simplicity, set $\eta(T) := \eta^v(T)$. Since

$$d(X^*H^* + H^*)R^{-1}Z = (X^*H^* + H^*)R^{-1}dz + (X^*F^* + v^*G^* + f^*)H^*R^{-1}Zdt + dWH^*R^{-1}Z + [X^*(H^*R^{-1})' + (H^*R^{-1})']Zdt$$

we have

$$\begin{aligned}\eta(T) = \exp\Big\{&(X^*(T)H^*(T) + H^*(T))R^{-1}(T)Z(T) \\ &- \int_0^T \Big[(X^*F^* + v^*G^* + f^*)H^*R^{-1}Z + (X^*(H^*R^{-1})' + (H^*R^{-1})')Z \\ &+ \frac{1}{2}(X^*H^* + H^*)R^{-1}(HX + H)\Big]dt \\ &- \int_0^T Z^*R^{-1}HdW\Big\}\end{aligned}$$

Formula (4.82) gives,

$$\begin{aligned}X_v = \theta \exp\Big\{&\int_0^T \big[\theta(X^*MX + v^*Nv + 2mX + 2nv) - (X^*F^* + v^*G^* + f^*)H^*R^{-1}Z \\ &-(X^*(H^*R^{-1})' + (H^*R^{-1})')Z - \frac{1}{2}(X^*H^* + H^*)R^{-1}(HX + H)\Big]dt - \int_0^T Z^*R^{-1}HdW\Big\} \\ &\times \exp\big\{\theta\big[X^*(T)M_TX(T) + 2m_TX(T) + (X^*(T)H^*(T) + H^*(T))R^{-1}(T)Z(T)\big]\big\}\end{aligned} \tag{4.88}$$

□

With this, we will proceed towards our next result by defining the function

$$p(x,t) = \exp\Big\{-\frac{1}{2}[X^*\Sigma(t)X - 2\sigma(t)X - \beta(t)]\Big\} \tag{4.89}$$

in which X is a parameter.
It is evident that,

$$p(X(T),T) = \exp\big\{\theta[X^*(T)M_TX(T) + 2m_TX(T)] + (X^*(T)H^*(T) + H^*(T))R^{-1}(T)Z(T)\big\} \tag{4.90}$$

Let also $\lambda(t)$ be defined by

$$\begin{aligned}\lambda(t) = \exp\Big\{&\int_0^t \big[\theta(X^*MX + v^*Nv + 2mX + 2nv) - (X^*F^* + v^*G^* + f^*)H^*R^{-1}Z \\ &-(X^*(H^*R^{-1})' + (H^*R^{-1})')Z - \frac{1}{2}(X^*H^* + H^*)R^{-1}(HX + H)\Big]ds - \int_0^t Z^*R^{-1}HdW\Big\}\end{aligned} \tag{4.91}$$

then from (4.88) we get

$$X_v = \theta\lambda(T)p(X(T),T) \tag{4.92}$$

Remark 4.1. *Observe that,*

- *The function $p(x,t)$ depends parametrically on $Z(\cdot)$.*

- *Since $Z(\cdot)$ and $X(\cdot)$ are independent, we can evaluate $p(X(T),T)$ by computing the Itô differential $dp(x(t),t)$, treating $Z(\cdot)$ as fixed.*
- *Itô's formula can be applied if $p(x,t)$ is continuously differentiable in time (it is already C^2 in x).*
- *This differentiability is not strictly satisfied due to the forms of $\dot{\sigma}$ and $\dot{\beta}$ from equations (4.85) and (4.86). However, the differentiability holds if the time-dependent coefficients $F, Q, (R^{-1}H)', m, f, G$ are continuous.*
- *There is an underlying assumption that control $v(\cdot)$ is a continuous process.*
- *The validity of equation (4.87) under slightly more regular assumptions implies it holds under less regular (i.e., L^∞) assumptions.*

Itô's formula, gives

$$dp(X(t),t) = p\Big\{-\frac{1}{2}(X_t^*\dot{\Sigma}_t X_t - 2\dot{\sigma}_t X_t + \dot{\beta}_t) - (X_t^*\Sigma_t - \sigma_t)(FX_t + f_t + G_t v_t) \\ +\frac{1}{2}(X_t^*\Sigma_t - \sigma_t)Q_t(\Sigma_t X_t - \sigma_t^*) - \frac{1}{2}\operatorname{tr}\Sigma_t Q_t - (X_t^*\Sigma_t - \sigma_t)dW_t\Big\}$$

It is easy to express $d\lambda(t)p(X(t),t)$ from above. Further, upon suitable choice of Σ, σ, β, one can see that

$$d(\lambda(t)p(X(t),t)) = -\lambda(t)p(X(t),t)[Z_t^*R^{-1}H_t + X_t^*\Sigma_t - \sigma_t]dW$$

and hence,

$$\lambda(t)p(X(t),t) = p(\xi,0)\exp\Big\{\int_0^t -(Z^*R^{-1}H + X^*\Sigma - \sigma)dW - \frac{1}{2}\int_0^t |H^*R^{-1}Z + \Sigma X - \sigma|^2 ds\Big\}$$

Then from (4.92) and the definition of $p(x,t)$, we get immediately (4.87). □

Lemma

We assume (4.83), (4.84) and

$$I + P_0^{1/2}\Sigma(0)P_0^{1/2} > 0 \tag{4.93}$$

then one has

$$\mathbb{E}[X_v|Z^T] = \frac{\theta}{|I + P_0^{1/2}\Sigma(0)P_0^{1/2}|^{1/2}} \\ \times \exp\Big\{-\frac{1}{2}[X_0^*\Sigma(0)X_0 - 2\sigma(0)X_0 + \beta(0)] \\ - (X_0^*\Sigma(0) - \sigma(0))P_0^{1/2}(I + P_0^{1/2}\Sigma(0)P_0^{1/2})^{-1} \\ \times P_0^{1/2}(\Sigma(0)X_0 - \sigma(0)^*)\Big\} \tag{4.94}$$

Proof. As we have $X_v > 0$ (up to the multiplicative constant θ), we can consider its expectation (or conditional expectation), with a possible infinite value.

Observing that,

$$\mathbb{E}\left\{\exp\left[\int_0^T -(Z^*R^{-1}H + X^*\Sigma - \sigma)dW - \frac{1}{2}\int_0^T |H^*R^{-1}Z + \Sigma X - \sigma|^2 dt\right]\middle| Z^T, \xi\right\} = 1 \quad \text{a.s.} \tag{4.95}$$

Since the processes $Z(\cdot)$, ξ, and $W(\cdot)$ are mutually independent, we evaluate equation (4.95) by treating $Z(\cdot)$ and ξ as fixed, and computing the expectation with respect to $W(\cdot)$ only.

The argument $-(Z^*R^{-1}H+X^*\Sigma-\sigma)$ can be written as the sum of $-(Z^*R^{-1}H+X_1^*\Sigma-\sigma)$ which is deterministic and $-\alpha^*\Sigma$ which is Gaussian.

From the definition of X_v, we deduce at once that

$$\mathbb{E}[X_v|Z^T,\xi] = \theta\exp\left\{-\frac{1}{2}[\xi^*\Sigma(0)\xi - 2\sigma(0)\xi + \beta(0)]\right\}.$$

We then compute the expectation of the right-hand side with respect to the Gaussian random variable ξ, treating $Z(\cdot)$ as fixed due to their independence. Because ξ has mean X_0 and covariance P_0, this expectation can be evaluated explicitly. The assumption in equation (4.93) ensures that this expectation is finite, and the result coincides with the right-hand side of equation (4.94).

□

An alternate expression for conditional expectation (4.94), can be derived as following.

We first introduce a new Riccati equation as follows:

$$\dot{P} - FP - PF^* + P(H^*R^{-1}H - 2\theta M)P - Q = 0, \quad P(0) = P_0 \tag{4.96}$$

and we shall assume

$$P^{-1}(t) + \Sigma(t) \geq \delta I, \quad \delta > 0, \quad \forall t \in [0,T] \tag{4.97}$$

We next introduce the processes

$$\begin{aligned} dr &= \left[(F - PH^*R^{-1}H + 2\theta PM)r + Gv + f + 2\theta Pm^* - PH^*R^{-1}H\right]dt + PH^*R^{-1}dz \\ r(0) &= x_0 \end{aligned} \tag{4.98}$$

$$\begin{aligned} s_t = {} & X_0^*P_0^{-1}x_0 + \log((2\pi)^n|P_0|) \\ & + \int_0^t \Big\{ \operatorname{tr}(P^{-1}Q + 2F) - r^*P^{-1}QP^{-1}r + 2r^*P^{-1}(Gv+f) \\ & - 2\theta(v^*Nv + 2nv) + 2z^*(R^{-1}H)' + H^*R^{-1}H\Big\}ds \end{aligned} \tag{4.99}$$

Justification of the above equations.

Note that

- $r(t), s(t)$ are forward processes,
- $\sigma(t), \beta(t)$ are backward processes.
- (4.94) can be read as

$$\mathbb{E}[X_v|Z^T] = \theta\int p(x,0)\Pi(x)dx \tag{4.100}$$

where

$$\Pi(x) = \frac{\exp\big(-\frac{1}{2}(x-x_0)^* P_0^{-1}(x-x_0)\big)}{(2\pi)^{n/2}\sqrt{|P_0|}} \tag{4.101}$$

represents the density of the initial random variable ξ.

Claim: There exists a process $q(x,t)$ (which is forward) such that

$$\int p(x,t)q(x,t)dx = \int p(x,0)q(x,0)dx, \quad \forall t \in [0,T] \tag{4.102}$$

and

$$q(x,0) = \Pi(x)$$

Proof. Observe that $p(x,t)$ is the solution of parabolic equation:

$$\begin{aligned} -\frac{\partial p}{\partial t} &= \frac{1}{2}\operatorname{tr}(D^2 pQ + Dp(Fx + Gv + f - QH^* R^{-1} z)) \\ &\quad + \frac{p}{2}\big\{-X^*(H^* R^{-1} H - 2\theta M)X + 2\theta(v^* N v + 2nv) \\ &\quad + 4\theta m X - 2X^*[R^{-1}H(FX + Gv + f) + (R^{-1}H)'X + (R^{-1}H)'] \\ &\quad -2H^* R^{-1} H X + z^* R^{-1} H Q H^* R^{-1} z - H^* R^{-1} H\big\} \end{aligned} \tag{4.103}$$

with

$$p(x,T) = \exp\big[\theta(X^* M_T X + 2m_T X) + (X^* H^*(T) + H^*(T))R^{-1}(T)Z(T)\big]$$

Therefore, to achieve (4.102), one can introduce the formal parabolic equation adjoint to (4.103), namely

$$\begin{aligned} \frac{\partial q}{\partial t} &= \frac{1}{2}\operatorname{tr}(D^2 qQ - Dq(FX + Gv + f - QH^* R^{-1} z)) \\ &\quad + \frac{q}{2}\big\{-X^*(H^* R^{-1} H - 2\theta M)X + 2\theta(v^* N v + 2nv) \\ &\quad + 4\theta m X - 2X^*[R^{-1}H(FX + Gv + f) + (R^{-1}H)'X + (R^{-1}H)'] \\ &\quad -2H^* R^{-1} H X + z^* R^{-1} H Q H^* R^{-1} z - 2\operatorname{tr} F - H^* R^{-1} H\big\} \end{aligned} \tag{4.104}$$

with initial condition:

$$q(x,0) = \Pi(x)$$

It is easy to shows that the process

$$q(x,t) = \exp\left\{-\frac{1}{2}\big[X^* P^{-1}(t)X - 2X^* P^{-1}(t)r(t)\big] - P(t)H^*(t)R^{-1}(t)Z(t) + s_t\right\} \tag{4.105}$$

is the solution of (4.104).

From (4.97), and after multiplying (4.104) by p, and performing integrations by parts in x, we get (4.102). □

Lemma

We assume (4.83), (4.84), (4.93), (4.97); then

$$\begin{aligned}\mathbb{E}[X_v|Z^T] = \theta \exp\Big\{\theta\Big[&r_T^*(I-2\theta M_T P(T))^{-1}(M_T r_T + 2m_T^*)\\ &+ 2\theta^2 m_T (P^{-1}(T) - 2\theta M_T)^{-1} m_T^*\\ &+ \int_0^T (r^*H^* + H^*)R^{-1}dz - \frac{1}{2}\int_0^T (r^*H^* + H^*)R^{-1}(Hr+H)dt\\ &+ \theta\int_0^T (r^*Mr + 2mr + v^*Nv + 2nv)dt + \theta\int_0^T \mathrm{tr}(PM)dt\Big]\Big\}\\ &\times |I - 2\theta M_T P(T)|^{-1/2}\end{aligned} \tag{4.106}$$

Proof. From (4.100) and (4.102) we get

$$\mathbb{E}[X_v|Z^T] = \theta \int p(x,T)q(x,T)dx.$$

Using (4.89), (4.85), (4.86), and (4.105), it follows that

$$\begin{aligned}\mathbb{E}[X_v|Z^T] = \theta \int \exp\Big\{&-\frac{1}{2}\big[X^*(P^{-1}(T) - 2\theta M_T)X - 2X^*(P^{-1}(T)r(T) + 2\theta m_T^*)\big]\\ &+ s_T - 2H^*(T)R^{-1}(T)Z(T)\Big\}dx\end{aligned} \tag{4.107}$$

It remains to prove that

$$\begin{aligned}s_T = &\, r_T^* P^{-1}(T) r_T + \log((2\pi)^n |P(T)|) + 2H^*(T)R^{-1}(T)Z(T)\\ &- 2\int_0^T (r^*H^* + H^*)R^{-1}dz + \int_0^T (r^*H^* + H^*)R^{-1}(Hr+H)ds\\ &- 2\theta\int_0^T (r^*Mr + 2mr + v^*Nv + 2nv)dt - 2\theta\int_0^T \mathrm{tr}(PM)dt\end{aligned} \tag{4.108}$$

which follows from a simple calculation, noting in particular that since the equation (4.96) can be written as

$$\dot{P} = (F + PF^*P^{-1} - PH^*R^{-1}H + 2\theta PM + QP^{-1})P$$

then one has the property

$$\begin{aligned}\frac{d}{dt}\log|P(t)| &= \mathrm{tr}\big(F + PF^*P^{-1} - PH^*R^{-1}H + 2\theta PM + QP^{-1}\big)\\ &= \mathrm{tr}\big(2F - PH^*R^{-1}H + QP^{-1} + 2\theta PM\big).\end{aligned}$$

RHS of (4.107) implies the desired result. □

4.3. Optimal control. Now, the the problem with partial observation is reducedfull observationtion under the assumptions of Lemma 4.2, the original problem (4.66), (4.67), (4.77). This is described by the following equation.

$$\begin{aligned}dr &= \big[(F - PH^*R^{-1}H + 2\theta PM)r + Gv + f + 2\theta Pm^* - PH^*R^{-1}H\big]dt + PH^*R^{-1}dz\\ r(t) &= x_0\end{aligned} \tag{4.109}$$

and the payoff is defined by

$$J(v(\cdot)) = \theta\mathbb{E}\exp\Big\{\theta\Big[r_T^*(I-2\theta M_T P(T))^{-1}(M_T r_T + 2m_T^*) + \int_0^T (r^*H^* + H^*)R^{-1}dz \\ -\frac{1}{2}\int_0^T (r^*H^* + H^*)R^{-1}(Hr+H)dt + \int_0^T (r^*Mr + 2mr + v^*Nv + 2nv)dt\Big]\Big\} \tag{4.110}$$

(with a possible value $+\infty$). The control $v(\cdot)$ belongs to $\mathcal{U}$.
Upon multiplying (4.110) by the constant,

$$\frac{\exp\Big[2\theta^2 m_T(P^{-1}(T)-2\theta M_T)^{-1}m_T^* + \theta\int_0^T \mathrm{tr}(PM)dt\Big]}{|I-2\theta M_T P(T)|^{1/2}}$$

we can recover our desired form (4.77).

We then use a similar method, as in (Section 2) to solve. Introduce, for any control $v(\cdot)$, the process

$$\varphi_t = \int_0^t (r^*H^* + H^*)R^{-1}dz - \frac{1}{2}\int_0^t (r^*H^* + H^*)R^{-1}(Hr+H)ds \\ + \theta\int_0^t (r^*Mr + 2mr + v^*Nv + 2nv)dt + \theta r_t^* S(t) r_t + 2\theta s_t^* r_t + \theta\rho_t \tag{4.111}$$

where $S(\cdot), s(\cdot), \rho(\cdot)$ are defined later. We require

$$S(T) = \frac{1}{2}\big[(I-2\theta M_T P(T))^{-1}M_T + M_T(I-2\theta P(T)M_T)^{-1}\big] \\ s(T) = m_T(I-2\theta P(T)M_T)^{-1} \\ \rho(T) = 0. \tag{4.112}$$

With the above definitions in mind, we obtain from (4.110) that

$$J(v(\cdot)) = \theta\mathbb{E}\exp\varphi_T \tag{4.113}$$

$$d\exp\varphi_t = \exp\varphi_t\Big\{\theta r_t^*[\dot{S}_t + S_t(F+2\theta PM) + (F^*+2\theta MP)S_t - S_t(GN^{-1}G^* \\ - 2\theta PH^*R^{-1}HP)S_t + M]r_t \\ + 2\theta[\dot{s}_t + s_t(F + 2\theta PH^*R^{-1}HPS + 2\theta PM - GN^{-1}G^*S) + m_t \\ + (f^* + 2\theta mP - nN^{-1}G^*)S_t]r_t \\ + \theta[\dot{\rho}_t + 2\theta s_t PH^*R^{-1}HPs_t - (s_tG_t + n_t)N_t^{-1}(G_t^*s_t^* + n_t^*) \\ + 2s_t(f_t + 2\theta P_t m_t^*) + \mathrm{tr}(PSP^*R^{-1}H)] \\ + \theta[v_t^* + (r_t^*S_t + s_t)G_tN_t^{-1} + n_tN_t^{-1}]N_t[v_t + N_t^{-1}G_t^*(S_tr_t + s_t^*) + N_t^{-1}n_t^*]\Big\}dt \\ + \exp\varphi_t[r_t^*(I+2\theta S_tP_t) + 2\theta s_tP_t]H^* + H^*\Big)R^{-1}dz$$

Assuming that the solution of the Riccati equation (4.114) (with initial condition (4.112)) exists on the interval $(0,T)$, fixing the $S(t), s(t), \rho(t)$ we have,

$$\dot{S}_t + S_t(F + 2\theta PM) + (F^* + 2\theta MP)S_t - S_t(GN^{-1}G^* - 2\theta PH^*R^{-1}HP)S_t + M = 0 \tag{4.114}$$

$$\dot{s}_t + s_t(F + 2\theta PH^*R^{-1}HPS + 2\theta PM - GN^{-1}G^*S) + m_t + (f^* + 2\theta mP - nN^{-1}G^*)S_t = 0 \tag{4.115}$$

$$\dot{\rho}_t + 2\theta PH^*R^{-1}HPs_t - (s_tG_t + n_t)N_t^{-1}(G_t^*s_t^* + n_t^*) + 2s_t(f_t + 2\theta P_t m_t^*) + \mathrm{tr}(PSP^*R^{-1}H) = 0 \tag{4.116}$$

which gives

$$\begin{aligned} d\exp\varphi_t = {} & \exp\varphi_t(r_t^*(I + 2\theta S_tP_t) + 2\theta s_tP_t)H^* + H^*\Big)R^{-1}dz \\ & + \theta\exp\varphi_t\big[v_t^* + (r_t^*S_t + s_t)G_tN_t^{-1} + n_tN_t^{-1}\big]N_t \\ & \times \big[v_t + N_t^{-1}G_t^*(S_tr_t + s_t^*) + N_t^{-1}n_t^*\big]dt, \end{aligned}$$

and, hence,

$$\begin{aligned} \theta\exp\varphi_T = {} & \theta\exp\varphi_0\exp\bigg\{\int_0^T [r_t^*(I + 2\theta S_tP_t) + 2\theta s_tP_t]H^* + H^*\bigg\}R^{-1}dz \\ & - \frac{1}{2}\int_0^T [r_t^*(I + 2\theta S_tP_t) + 2\theta s_tP_t]H^* + H^*\Big)R^{-1} \\ & \times H[(I + 2\theta P_tS_t)r_t + 2\theta P_ts_t^* + h]dt \\ & \times \exp\theta\int_0^T \big[v_t^* + (r_t^*S_t + s_t)G_tN_t^{-1} + n_tN_t^{-1}\big] \\ & \times N_t\big[v_t + N_t^{-1}G_t^*(S_tr_t + s_t^*) + N_t^{-1}n_t^*\big]dt. \end{aligned} \tag{4.117}$$

Thus, for

$$v(\cdot) \in \mathcal{U}, \tag{4.118}$$

and

$$\begin{aligned} & \mathbb{E}\exp\bigg(\int_0^T [r_t^*(I + 2\theta S_tP_t) + 2\theta s_tP_t]H^* + H^*\Big)R^{-1}dz \\ & -\frac{1}{2}\int_0^T [r_t^*(I + 2\theta S_tP_t) + 2\theta s_tP_t]H^* + H^*\Big)R^{-1}H[(I + 2\theta P_tS_t)r_t + 2\theta P_ts_t^* + h]dt\bigg) = 1 \end{aligned}$$

We define a particular control denoted by $u(t)$ such that

$$\boxed{u_t = -N_t^{-1}G_t^*(S_t\hat{r}_t + s_t^*) - N_t^{-1}n_t^*} \tag{4.119}$$

with the process $\hat{r}_t$ defined by

$$\begin{aligned} d\hat{r} = {} & \Big[(F - PH^*R^{-1}H + 2\theta PM - GN^{-1}G^*S)\hat{r} - PH^*R^{-1}h \\ & - GN^{-1}(G^*s^* + n^*) + f + 2\theta Pm^*\Big]dt + PH^*R^{-1}dz \\ \hat{r}(0) = {} & x_0 \end{aligned} \tag{4.120}$$

We can then state our main result below.

Theorem 4.1

We assume (4.65), (4.83). We also assume that the Riccati equations (4.84), (4.96), (4.114) have a global solution on $[0,T]$ such that (4.97) holds true. Consider the class of admissible controls defined by (4.118) and the payoff (4.110) (or (4.77), which is the same up to a positive multiplicative constant). Then the control defined by formulas (4.119), (4.120) is optimal.

Proof. From (4.117) and (4.113) we see that, for $v(\cdot)$ belongs to the class of admissible controls, we have

$$J(v(\cdot)) \geq \theta \exp \varphi_0. \tag{4.121}$$

Moreover, for the control $u(\cdot)$, the corresponding process φ, denoted by $\hat{\varphi}$, satisfies

$$\begin{aligned}\theta \exp \hat{\varphi}_T = \theta \exp \varphi_0 \exp\Bigg(\int_0^T [\hat{r}_t^*(I + 2\theta S_t P_t) + 2\theta s_t P_t] H^* + H^* \Bigg) R^{-1} dz \\ -\frac{1}{2} \int_0^T \hat{r}_t [\hat{r}_t^*(I + 2\theta S_t P_t) + 2\theta s_t P_t] H^* + H^* \Bigg) R^{-1} \\ \times H[(I + 2\theta P_t S_t)\hat{r}_t + 2\theta P_t s_t^* + h] dt\end{aligned} \tag{4.122}$$

Since $\hat{r}(t)$ is a Gaussian process, and by the identity established in (2.17) (referencing the main result from the full-observation case), the expectation of the second exponential term in (4.122) equals 1. Therefore, we obtain:

$$\mathbb{E}\theta \exp \hat{\varphi}_T = \theta \exp \varphi_0.$$

This proves at the same time that $u(\cdot)$ is admissible and optimal. □

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