

Taylor's Inequality

The remainder of the Taylor series for $f(x)$ centered at a is defined as

$$R_n(x) = f(x) - T_n(x),$$

where $T_n(x)$ denotes the n th degree Taylor polynomial (also centered at a). Taylor's Inequality says that if $|f^{(n+1)}(x)| \leq M$ for all x satisfying $|x - a| \leq d$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1} \quad \text{for } x \text{ satisfying } |x - a| \leq d.$$

1. Use Taylor's inequality to determine the number of terms needed to estimate $\cos(\frac{1}{2})$ using the Taylor series centered at $a = \frac{\pi}{6}$ to within $0.0001 = 10^{-4}$.

← looking for n

tells us $f(x) = \cos x$
and $x = 1/2$

Given: $a = \pi/6$

$$f(x) = \cos x$$

$x = 1/2$ ← the point we want to approximate at

$$\text{error} \leq 10^{-4}$$

Need: M so that $|f^{(n+1)}(x)| \leq M$

n so that the error is within the desired bound

First determine M : Since $f(x) = \cos x$, $f^{(n+1)}(x) = \pm \sin x / \pm \cos x$

$$\Rightarrow |f^{(n+1)}(x)| \leq 1 \quad \text{for all } n \text{ and all } x$$

Choose $M = 1$

Now plug in to Taylor's Inequality: $|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1}$

$$|R_n(1/2)| \leq \frac{1}{(n+1)!} \left| \frac{1}{2} - \frac{\pi}{6} \right|^{n+1} \leq 10^{-4} \quad \leftarrow \text{not solvable by hand so we make a table}$$

↑ set less than desired error

n	$\frac{1}{(n+1)!} \left \frac{1}{2} - \frac{\pi}{6} \right ^{n+1}$
1	0.0002784...
2	0.0000021...

← small enough so take $n = 2$

Answer: $n = 2$

Taylor's Inequality

The remainder of the Taylor series for $f(x)$ centered at a is defined as

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where $T_n(x)$ denotes the n th degree Taylor polynomial (also centered at a). Taylor's Inequality says that if $|f^{(n+1)}(x)| \leq M$ for all x satisfying $|x - a| \leq d$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1} \quad \text{for } x \text{ satisfying } |x - a| \leq d.$$

1. Use Taylor's Inequality to determine what degree Taylor polynomial centered at $a = \frac{\pi}{6}$ is needed to estimate $\cos(\frac{1}{2})$ to within $0.0001 = 10^{-4}$. ← looking for n

tells us $f(x) = \cos x$
and $x = 1/2$

Given: $a = \pi/6$

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Answer: $n = 2$

$\Rightarrow n=2$

2. (a) Find the 2nd degree Taylor polynomial, $T_2(x)$ for $f(x) = x^{-2}$ based at $a = 1$.
 (b) Use Taylor's Inequality to estimate the accuracy of the approximation using $T_2(x)$ to estimate $f(x)$ in the interval $[0.9, 1.1]$.

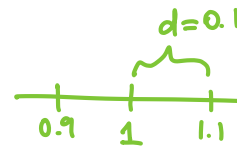
$\nwarrow$ largest error guaranteed by Taylor's Inequality

$$\begin{array}{lll}
 (a) \quad f(x) = x^{-2} & f(1) = 1 & C_0 = 1 \\
 f'(x) = -2x^{-3} & f'(1) = -2 & C_1 = \frac{-2}{1!} = -2 \\
 f''(x) = 6x^{-4} & f''(1) = 6 & C_2 = \frac{6}{2!} = 3 \\
 f'''(x) = -24x^{-5}
 \end{array}$$

$$T_2(x) = C_0 + C_1(x-1) + C_2(x-1)^2 = 1 - 2(x-1) + 3(x-1)^2$$

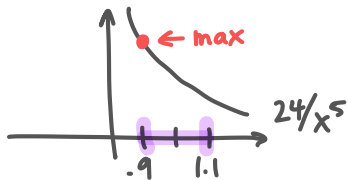
(b) Given: $f(x) = x^{-2}$
 $a = 1$
 $n = 2$

x in $[0.9, 1.1]$ so $d = 0.1$



Need: Max error for x in $[0.9, 1.1]$ according to Taylor's Inequality
 M so that $|f'''(x)| \leq M$ on $[0.9, 1.1]$ $\leftarrow$ since $n=2$

First choose M : $|f'''(x)| = \left| -\frac{24}{x^5} \right| = \frac{24}{x^5}$ on $[0.9, 1.1]$



Notice $24/x^5$ is decreasing on the interval $[0.9, 1.1]$ so max occurs at the left endpoint.

Choose $M = \frac{24}{(0.9)^5}$

Now plug in to Taylor's Inequality: $|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$

$$|R_2(x)| \leq \frac{24(0.9)^{-5}}{3!} \underbrace{|x-1|^3}_{\leq 0.1 \text{ if } x \text{ is in } [0.9, 1.1]} \leq \frac{24(0.9)^{-5}}{3!} (0.1)^3 \approx 0.00677$$

Answer
 ≈ 0.00677

$\Rightarrow$ want n $\rightarrow a=0$

3. How many terms of the Maclaurin series for $f(x) = \cos(2x)$ are needed to guarantee that the approximation of $f(x)$ on the interval $[-\frac{1}{3}, \frac{1}{3}]$ is within 10^{-6} ?

Given: $a=0$

$$f(x) = \cos(2x)$$

$$d = \frac{1}{3} \text{ (since the interval is } [-\frac{1}{3}, \frac{1}{3}])$$

Want: n so that $|\text{error}| \leq 10^{-6}$

$$M \text{ so that } |f^{(n+1)}(x)| \leq M \text{ for } x \text{ in } [-\frac{1}{3}, \frac{1}{3}]$$

Determine M: $f(x) = \cos(2x)$

$$f'(x) = -2 \sin(2x)$$

$$f''(x) = -2^2 \cos(2x)$$

$$f'''(x) = 2^3 \sin(2x)$$

$$f^{(4)}(x) = 2^4 \cos(2x)$$

 $\vdots$

$$f^{(n)}(x) = \pm 2^n \sin/\cos(2x)$$

$$\text{So } |f^{(n+1)}(x)| = |2^{n+1} \sin/\cos(2x)| \leq 2^{n+1} = M$$

Plug in: $|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1} \leq \frac{2^{n+1}}{(n+1)!} |x-0|^{n+1}$

$$\leq \frac{2^{n+1}}{(n+1)!} \left(\frac{1}{3}\right)^{n+1} \leq 10^{-6}$$

n	$\frac{1}{(n+1)!} \left(\frac{2}{3}\right)^{n+1}$
4	0.00109739
5	0.00012193
6	0.00001161
7	0.00000096 ✓

Answer: $n = 7^*$

* Small technicality: This question asks for the number of terms of the Maclaurin series.

When we say $n=7$, that is the degree of the Taylor polynomial that we should use to do the approximation

using the
Maclaurin series
for $\cos(x)$

$$\begin{aligned}\cos(2x) &= \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{4^n x^{2n}}{(2n)!} \\ &= 1 - \frac{4}{2!} x^2 + \frac{4^2}{4!} x^4 - \frac{4^3}{6!} x^6 + \frac{4^4}{8!} x^8 - \dots\end{aligned}$$

only use up to degree 7, which is the first 4 terms of the Maclaurin series for $\cos(2x)$

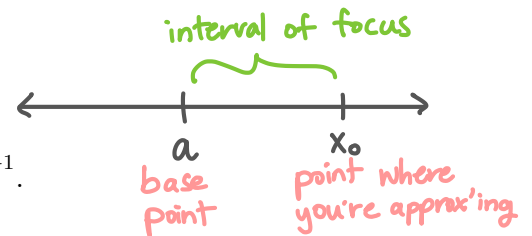
Real answer: 4 terms

If you're using a Taylor polynomial to approximate f at a particular value of x instead of a range of x , the following formulation of Taylor's Inequality is sometimes useful.

Alternate Statement of Taylor's Inequality

If $|f^{(n+1)}(x)| \leq M$ for all x between a and x_0 , then

$$|R_n(x_0)| \leq \frac{M}{(n+1)!} |x_0 - a|^{n+1}.$$

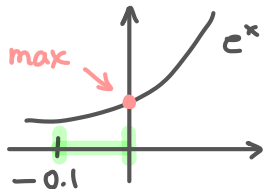


4. How large should n be to guarantee that the approximation of $e^{-0.1}$ using an n th degree Taylor polynomial for e^x centered at $a = 0$ is within 0.0001? $= 10^{-4}$
 $\Rightarrow x_0 = -0.1$

Given: $f(x) = e^x$
 $a = 0$
 $x_0 = -0.1$
 $\text{error} \leq 10^{-4}$

Need: M so that $|f^{(n+1)}(x)| \leq M$ for x in $[-0.1, 0]$
 n so that $|R_n(-0.1)| \leq 10^{-4}$

First choose M : Since $f(x) = e^x$, $f^{(n+1)}(x) = e^x$ for all n



Since e^x is increasing on $[-0.1, 0]$ we know that the max occurs at the right endpoint.

Choose $M = e^0 = 1$

Now plug in to Alt. Taylor's Ineq: $|R_n(x_0)| \leq \frac{M}{(n+1)!} |x_0 - a|^{n+1}$

$$|R_n(-0.1)| \leq \frac{1}{(n+1)!} |-0.1 - 0|^{n+1} = \frac{(0.1)^{n+1}}{(n+1)!} \leq 10^{-4}$$

can't solve by hand so table

n	$(0.1)^{n+1} / (n+1)!$
1	0.005
2	0.000166
3	0.00000416

$\leftarrow$ less than 10^{-4}

Answer: $n = 3$

looking for d since
 x is in $[a-d, a+d]$

5. Estimate the range of values of x for which the approximation

$$\ln(x) \approx \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2$$

← powers of $(x-2)$
 so $a=2$

is accurate to within 0.01.

degree 2 polynomial so $n=2$

Given: $f(x) = \ln(x)$

$$n=2$$

$$a=2$$

$$\text{error} \leq 0.01$$

Need: M so that $|f'''(x)| \leq M$ for x in $[2-d, 2+d]$

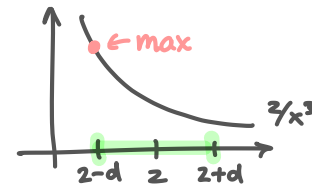
d so that $|R_2(x)| \leq 0.01$ for x in $[2-d, 2+d]$

First choose M : $f(x) = \ln x$

$$f'(x) = \frac{1}{x}$$

$$f''(x) = -x^{-2}$$

$$f'''(x) = 2x^{-3} = \frac{2}{x^3}$$



Choose $M = \frac{2}{(2-d)^3}$

Plug in to Taylor's Inequality: $|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$
 where $|x-a| \leq d$

(Since we're solving for d , replace $|x-a|$ with d)

$$\begin{aligned} |R_2(x)| &\leq \frac{2/(2-d)^3}{3!} \underbrace{|x-2|^3}_{\leq d} \leq \frac{1}{3!} \cdot \frac{2}{(2-d)^3} \cdot d^3 \\ &= \frac{d^3}{3(2-d)^3} \leq 0.01 \\ &\quad \text{true for } d = 0.474\dots \end{aligned}$$

Answer: $[1.526, 2.474]$