The Invertible Matrix Theorem. Let A be an $n \times n$ matrix, in other words A is a **square** matrix. The following are equivalent*:

(a) A is invertible

(b) $A \sim I$ (i.e. A is row equivalent to the $n \times n$ identity matrix)

(c) A has n pivot positions

(d) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

(e) The columns of A are linearly independent

(f) The columns of A span $\mathbb{R}^n$

(g) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one

(h) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto

(i) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $\mathbf{b}$ in $\mathbb{R}^n$

(j) There exists an $n \times n$ matrix C such that $CA = I$ (i.e. A has a left inverse)

(k) There exists an $n \times n$ matrix D such that $AD = I$ (i.e. A has a right inverse)

(l) A^T is invertible

(m) The columns of A form a basis of $\mathbb{R}^n$

(n) $\text{Col}(A) = \mathbb{R}^n$

(o) $\text{Nul}(A) = \{\mathbf{0}\}$

(p) $\text{rank}(A) = n$

(q) $\text{nullity}(A) = 0$

(r) $\det(A) \neq 0$

(s) 0 is not an eigenvalue of A

*Equivalent means either all the statements are true about a matrix A or all the statements are false.

Theorem 10 (*Linear Transformations that are One-to-One*)**.**
Suppose A is an $m \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^m$ is the associated linear transformation, $T(\mathbf{x}) = A\mathbf{x}$. The following are equivalent:

(a) T is one-to-one

(b) The range of T has dimension n

(c) For each $\mathbf{b} \in \mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ either has a unique solution or is inconsistent

(d) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

(e) The columns of A are linearly independent

(f) A has a pivot position in every column

(g) $\text{rank}(A) = n$

Theorem 11 (*Linear Transformations that are Onto*)**.**
Suppose A is an $m \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^m$ is the associated linear transformation, $T(\mathbf{x}) = A\mathbf{x}$. The following are equivalent:

(a) T is onto

(b) The range of T is equal to the entire codomain ($\text{range}(T) = \mathbb{R}^m$)

(c) $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^m$

(d) The columns of A span $\mathbb{R}^m$

(e) A has a pivot position in every row

(f) $\text{rank}(A) = m$