

Study on Systems of Nonlinear Conservation Laws Arising in Chemotaxis and Traffic Flow

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Chemotaxis System

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where (u_-, v_-) and (u_+, v_+) are constant states.

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- ⑤ (1) is a special case of (4) with $\epsilon = 0, D = 0$.

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- In our system (1), $U = (u, v)^T$ and $F(U) = (uv, -u)^T$.

Preliminaries

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Eigenvalues and Eigenvectors

$$\lambda_{1,2} = \frac{v \mp \sqrt{v^2 - 4u}}{2} \quad (7)$$

$$r_{1,2} = \frac{1}{2}(-v \mp \sqrt{v^2 - 4u}, 1)^T \quad (8)$$

Regions

$$H = \{(u, v) | u \geq 0, v^2 - 4u > 0\},$$

$$E = \{(u, v) | u \geq 0, v^2 - 4u < 0\}, \quad (9)$$

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Axis Regions

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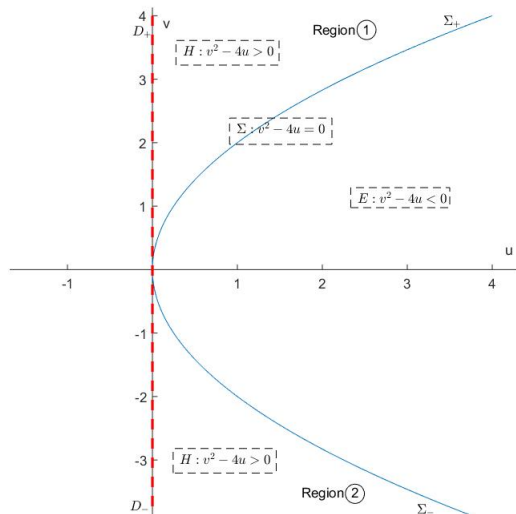
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- An eigenvalue λ_i is said to be linearly degenerate (LD) if $\nabla \lambda_i \cdot r_i = 0$. Eigenvalues are linearly degenerate on some portions of D .

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- ④ This work was published in [1].

Shock Curves

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- The Rankine-Hugoniot conditions applied to (1) yields

$$\begin{aligned} s(u - u_-) &= uv - u_- v_- \\ s(v - v_-) &= -(u - u_-) \end{aligned} \tag{11}$$

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$$s < \lambda_1(U_-), \quad \lambda_1(U_+) < s < \lambda_2(U_+).\tag{12}$$

Similarly, the 2-shocks satisfy

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$$\begin{aligned} S_1(U_-) : u - u_- &= -(v - v_-) \frac{v - \sqrt{v^2 - 4u_-}}{2}, \quad v > v_- \\ S_2(U_-) : u - u_- &= -(v - v_-) \frac{v + \sqrt{v^2 - 4u_-}}{2}, \quad v < v_- \end{aligned} \quad (14)$$

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Rarefaction

$$\begin{aligned} R_1(U_-) : \frac{du}{dv} &= \frac{-v + \sqrt{v^2 - 4u}}{2}, v < v_- \\ R_2(U_-) : \frac{du}{dv} &= \frac{-v - \sqrt{v^2 - 4u}}{2}, v > v_-. \end{aligned} \tag{15}$$

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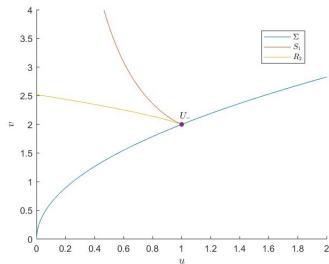
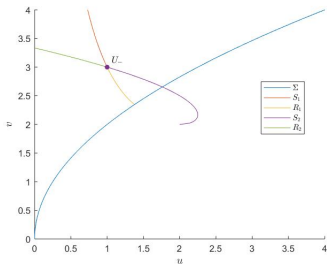
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- $R_i(U_-) = \{U \in \mathbb{R}^2 | w_i(U) = w_i(U_-)\}, i = 1, 2$
- Following Rascle and Zheng [5, 6]

$$\begin{aligned} w_1(U) &= \begin{cases} \sqrt{\lambda_1}(\lambda_1 + 3\lambda_2), & v > 0 \\ \sqrt{-\lambda_1}(\lambda_1 + 3\lambda_2), & v < 0. \end{cases} \\ w_2(U) &= \begin{cases} \sqrt{\lambda_2}(3\lambda_1 + \lambda_2), & v > 0 \\ \sqrt{-\lambda_2}(3\lambda_1 + \lambda_2), & v < 0. \end{cases} \end{aligned} \tag{16}$$

Shock and Rarefaction Curves



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Linearly Degenerate Regions: $\nabla \lambda_i \cdot r_i = 0$

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- Let $U_-, U_+ \in D_- \cup D_+$. Then,
- U_- and U_+ can be connected a contact discontinuity of speed $s = 0$.

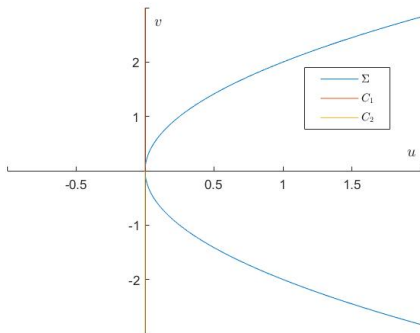


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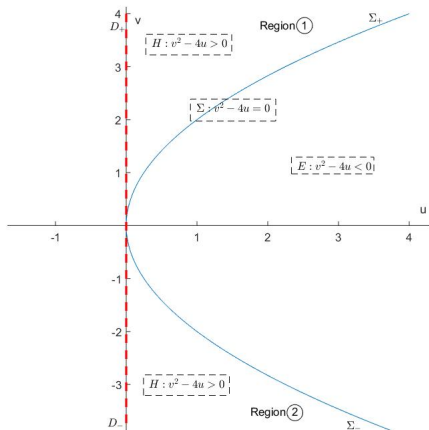
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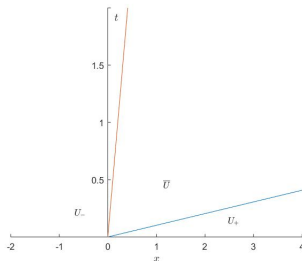
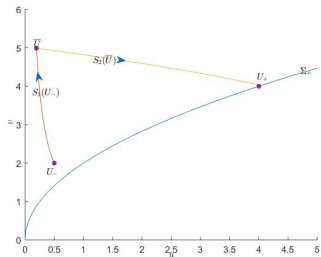
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- We find this intermediate state $\bar{U}$ by finding where the 1-wave from U_- intersects the 2-wave from U_+ , see [1], Section 6.1.
- We will show this via showcases.

Regions



$U_- = (0.5, 2)$, $U_+ = (4, 4) \in \Sigma_+$ and $\bar{U} \approx (0.194, 4.99)$.



$U_- = (4, -4) \in \Sigma_-$, $U_+ = (0.5, -2)$ and $\bar{U} \approx (0.19, -5)$.

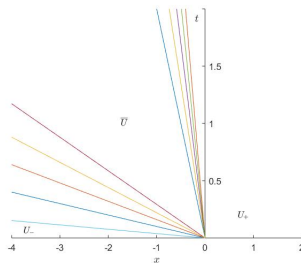
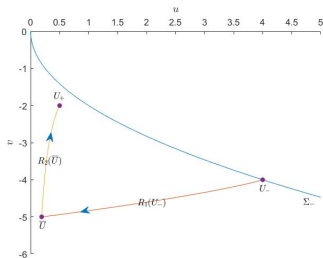


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- Viscous models studied by Kerner-Konhauser, Kuhne, Beckshulte, Li [12, 13, 14]

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- Aw-Rascle and Zhang's higher continuum (ARZ) models [15, 16]

System of Balance Laws

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- Consider the general form of balance laws

$$U_t + F(U)_x + P(U) = 0 \quad (17)$$

with initial data

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- Framework of Dafermos [4]
- He assumed $P(0) = 0$.

Traffic Flow Model

$$\begin{aligned}\rho_t + (\rho v)_x &= 0, \\ v_t + \left(\frac{1}{2}v^2 + g(\rho)\right)_x + \frac{v - v_e(\rho)}{\tau} &= 0,\end{aligned}\tag{19}$$

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- ① ρ - density, v - velocity, $v_e(\rho)$ - equilibrium velocity.
- ② $g(\rho)$ - anticipation factor and satisfies

$$g'(\rho) = \rho(v_e'(\rho)/\theta)^2,\tag{21}$$

where $g'(\rho) \geq 0$, and

$$0 < \theta < 1.\tag{22}$$

LWR Model

- The equilibrium flow is described by Lighthill-Whitham-Richards (LWR) model [18, 19]

$$\rho_t + (\rho v_e(\rho))_x = 0, \quad x \in \mathbb{R}, t > 0, \quad (23)$$

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- $q(\rho) = \rho v_e(\rho)$ is known as the fundamental diagram.
- For this study, we let $v_e(\rho)$ such that

$$v_e(0) = b, v'_e(0) = -a, v_e(1) = 0, \quad (24)$$

where $a, b > 0$ and $v'_e(\rho) < 0$.

Global BV Solution to a system of balance laws with *nonconcave* flux

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- Context - Nonconcave fundamental diagrams

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- Part of our work was published in [2] and [3].

First Transformation

Change of Variables

$$U = (\bar{\rho}, \bar{v}) = (\rho - \rho_*, v - v_*), \quad (25)$$

where $v_* = v_e(\rho_*)$. We show existence of global BV solutions for (19) near equilibrium states $(\rho, v) = (\rho_*, v_e(\rho_*))$ for all $\rho_* > 0$. With this transformation, $P(0) = 0$.

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Transformed System

$$\begin{aligned} \bar{\rho}_t + (\bar{\rho} + \rho_*)(\bar{v} + v_*)_x &= 0 \\ \bar{v}_t + \left(\frac{1}{2}(\bar{v} + v_*)^2 + g(\bar{\rho} + \rho_*)\right)_x + \frac{v + v_* - v_e(\bar{\rho} + \rho_*)}{\tau} &= 0 \end{aligned} \quad (26)$$

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$$\begin{aligned} F(U) &= (\bar{\rho}\bar{v} + v_*\bar{\rho} + \rho_*\bar{v}, \frac{1}{2}\bar{v}^2 + v_*\bar{v} + g(\bar{\rho} + \rho_*))^T \\ P(U) &= (0, \frac{\bar{v} + v_* - v_e(\bar{\rho} + \rho_*)}{\tau})^T. \end{aligned} \quad (27)$$

Eigenvalues

$$\lambda_{1,2} = \bar{v} + v_* \pm (\bar{\rho} + \rho_*) \frac{v'_e(\bar{\rho} + \rho_*)}{\theta}. \quad (28)$$

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- We do an *a priori* estimate: $TV(\bar{\rho}_0)$ is small (60) and (59) implies $TV(\bar{\rho})$ is small (63) and (62), which implies $\bar{\rho}$ is small.
- Since $\rho_* > 0$ and $\bar{\rho}$ is small by choosing $\bar{\rho}_0$, then $\bar{\rho} + \rho_* > 0$.

Entropy-Entropy Flux Pair

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- Admissible solutions U must satisfy the entropy inequality

$$\partial_t \eta(U(x, t)) + \partial_x q(U(x, t)) + D\eta(U(x, t))P(U(x, t)) \leq 0 \quad (29)$$

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- Solve for nontrivial companion balance laws

$$\begin{aligned} DQ_1(U) &= B_1(U)^T DG_1(U) \\ DQ_2(U) &= B_2(U)^T DG_2(U), \end{aligned} \quad (30)$$

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- $G_1 = U, G_2 = F(U), DQ_i = [\frac{\partial Q_i}{\partial \rho}, \frac{\partial Q_i}{\partial u}], i = 1, 2.$

Simplifying

- We calculate

$$Q_{1,\bar{\nu}\bar{\nu}} - s^2(\bar{\rho} + \rho_*)Q_{1,\bar{\rho}\bar{\rho}} = 0, \quad (31)$$

where

$$s(\bar{\rho} + \rho_*) = -\frac{\theta}{v'_e(\bar{\rho} + \rho_*)} > 0 \quad (32)$$

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- $\hat{q}(\bar{\rho} + \rho_*)$ in (34) is an actual fundamental diagram observed in traffic flow, see [7]. The system (26) is GNL under (33).

Concave Fundamental Diagram

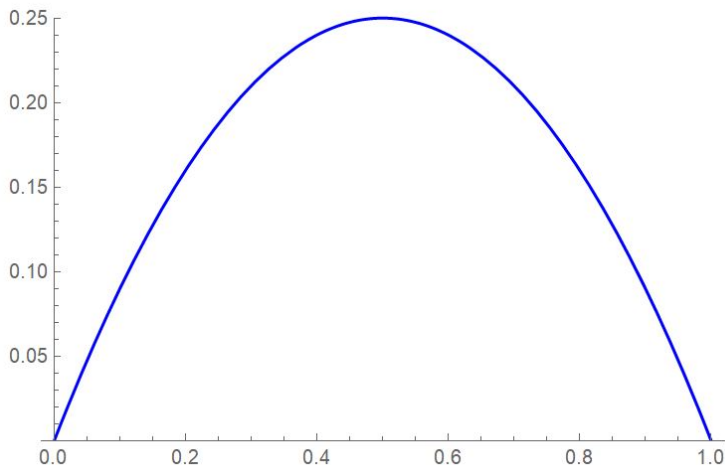


Figure: An Example of concave fundamental diagram

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$$\hat{\eta}(\bar{\rho}, \bar{v}) = \hat{Q}_1(\bar{\rho}, \bar{v}) = \left(n_1(\bar{v} + v_* + \frac{\hat{v}_e(\bar{\rho} + \rho_*)}{\theta}) \right)^2 + \left(n_2(\bar{v} + v_* - \frac{\hat{v}_e(\bar{\rho} + \rho_*)}{\theta}) \right)^2, \quad (35)$$

where $n_1, n_2 > 0$.

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$$\hat{q}(\bar{\rho}, \bar{v}) = \hat{Q}_2(\bar{\rho}, \bar{v}) = \hat{Q}_1(\bar{\rho}, \bar{v})(\bar{v} + v_*) + \int \hat{Q}_{1,\bar{v}}(\bar{\rho} + \rho_*) \left(\frac{\hat{v}'_e(\bar{\rho} + \rho_*)}{\theta} \right)^2 d\bar{\rho} + j(\bar{v}). \quad (36)$$

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- Finally, we can estimate

$$Q_1(\rho, u) = \hat{Q}_1(\rho, u) + \mathcal{O}(\gamma), \quad (38)$$

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Nonconcave Fundamental Diagram

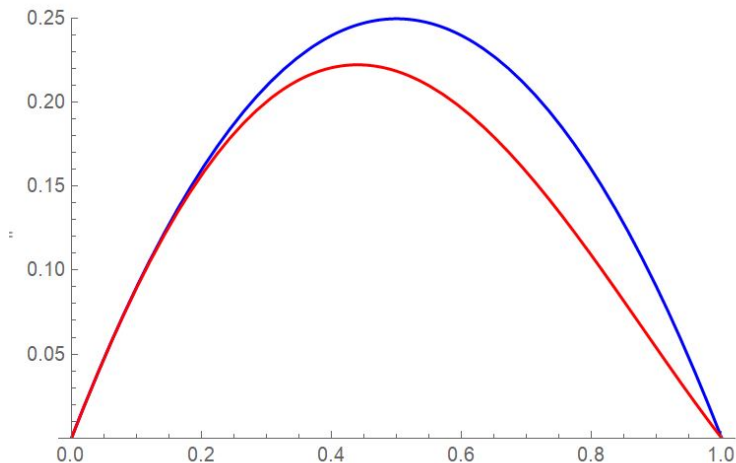


Figure: An Example of non-concave fundamental diagram

Convexity Conditions

$$Q_{1,\bar{\rho}\bar{\rho}} > 0, \quad (40)$$

$$\mathcal{D} = \det \begin{bmatrix} Q_{1,\bar{\rho}\bar{\rho}} & Q_{1,\bar{\rho}\bar{v}} \\ Q_{1,\bar{v}\bar{\rho}} & Q_{1,\bar{v}\bar{v}} \end{bmatrix} > 0. \quad (41)$$

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$$\hat{Q}_{1,\bar{\rho}\bar{\rho}} = \left(\frac{\hat{v}'_e(\bar{\rho} + \rho_*)}{\theta} \right)^2 \left(\hat{\beta}'' n_1^2 + \hat{G}'' n_2^2 \right) > 0. \quad (42)$$

$$\hat{\mathcal{D}} = 4 \left(\frac{\hat{v}'_e(\bar{\rho} + \rho_*)}{\theta} \right)^2 \hat{\beta}'' \hat{G}'' n_1^2 n_2^2 > 0, \quad (43)$$

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- ① For $\gamma > 0$ small, $Q_1(\bar{\rho}, \bar{v})$ satisfies (40) and (41).

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- We assume that P is dissipative semidefinite relative to η , i.e.

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- For our system (26), we needed to find a condition such that

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- We require

$$0 < \alpha \leq 2\zeta\tau, \quad (46)$$

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- For our system, we have

$$DP(0)r_i(0) = \begin{bmatrix} 0 \\ \frac{\mp\theta + 1}{\tau} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (49)$$

since $0 < \theta < 1$.

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- The sub-characteristic condition is satisfied for (26) since we have

$$\bar{v} + v_* - \frac{a}{\theta}(\bar{\rho} + \rho_*) < -2a(\bar{\rho} + \rho_*) + b < \bar{v} + v_* + \frac{a}{\theta}(\bar{\rho} + \rho_*) \quad (51)$$

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- In order to apply Dafermos' theory, we needed to convert (26) into an equivalent form

$$\begin{aligned}\partial_t V + \partial_x G(V, W) &= 0 \\ \partial_t W + \partial_x H(V, W) + C(V, W)W &= 0,\end{aligned}\tag{52}$$

where $x \in \mathbb{R}$, $t > 0$, and $\eta_{WW}C(0, 0) > 0$.

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- We calculate

$$DP(0) = \frac{1}{\tau} \begin{bmatrix} 0 & 0 \\ -v'_e(\rho_*) & 1 \end{bmatrix} = \frac{1}{\tau} \begin{bmatrix} 0 & 0 \\ a_* & 1 \end{bmatrix} = S^{-1} \Gamma S,$$

where $S = \begin{bmatrix} 1 & 0 \\ a_* & 1 \end{bmatrix}$ and $\Gamma = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\tau} \end{bmatrix}$.

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New System

This transformed (26) to

$$V_t + ((V + \rho_*)(W - a_* V + v_*))_x = 0, \quad (55)$$

$$W_t + \left(\frac{1}{2} (W^2 - a_*^2 V^2) + a_* \rho_* (W - a_* V) + v_* W + g(V + \rho_*) \right)_x \\ + C(V, W)W = 0. \quad (56)$$

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- Let

$$Z = (V, W) = (\bar{\rho}, a\bar{\rho} + \bar{v}). \quad (57)$$

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- W in (54) is the difference between velocity and equilibrium velocity
- Significance of transforming is to identify and expose the component of the state vector that is directly affected by damping.

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Theorem 2

Under the conditions (21) (22) (37), the system of balance laws (55) (56) is endowed with a convex entropy-entropy flux pair (η, q) , satisfies the partial dissipative inequality, and satisfies the Kawashima condition. Then there exists positive constants δ_0, σ_0 such that if Z_0 satisfies

$$\int_{-\infty}^{\infty} (x^2 + 1) |Z_0(x)|^2 dx = \sigma^2 < \sigma_0^2, \quad (59)$$

with bounded variation

$$TV_{(-\infty, \infty)} Z_0(\cdot) = \delta < \delta_0, \quad (60)$$

and

$$\int_{-\infty}^{\infty} V_0(x) dx = 0, \quad (61)$$

Theorem Continued

then the Cauchy problem (55) (56) (58) possesses a unique admissible *BV* solution Z defined on $(-\infty, \infty) \times [0, \infty)$ and

$$\int_{-\infty}^{\infty} |Z(x, t)| \, dx \leq c_0 \sigma, \quad 0 \leq t < \infty, \quad (62)$$

$$TV_{(-\infty, \infty)} Z(\cdot, t) \leq c_1 \sigma + c_2 \delta e^{-\nu t}, \quad 0 \leq t < \infty, \quad (63)$$

$$\int_{-\infty}^{\infty} |Z(x, t)| \, dx \rightarrow 0, \quad \text{as } t \rightarrow \infty, \quad (64)$$

$$TV_{(-\infty, \infty)} Z(\cdot, t) \rightarrow 0, \quad \text{as } t \rightarrow \infty \quad (65)$$

for some positive constants c_0, c_1, c_2, ν .

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Future Work

Future Work

- Studying occurrence of transitional waves

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- Studying occurrence of transitional waves
- Analyzing viscous profiles in mixed-type problems

Future Work

- Studying occurrence of transitional waves
- Analyzing viscous profiles in mixed-type problems
- Generalizing to nonconcave fundamental diagrams with large perturbations

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The End

- Thank You!
- Questions?